

MinPROMEP: Generation of Partially Replicated Minimal Orthogonal Main-effect Plans Using a Novel Algorithm

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Abstract

MinPROMEP, written in SAS macros, is a program that generates partially replicated minimal orthogonal main-effect plans (minimal PROMEP's) for a given set of specified factor levels. The proposed algorithm is based on a generalization of the construction of Jacroux (1992, 1993). By using specific collapsing schemes, duplicated points are embedded in the design. In some cases, the generated minimal PROMEP has maximum number of duplicate points.

1 Introduction

Orthogonal main-effect plans (OMEF's) are effectively applied in the early stage in experiments to locate significant factors in many applications. As stated by Lewis and John (1976), orthogonality stands for the uncorrelated property of the estimates of all main effects in the plan while testing for hypotheses of weighted means. Under such considerations, proportional frequencies are both sufficient and necessary conditions of an OMEF according to Plackett (1946) and Addelman (1962). As noted by Jacroux (1992), partially replicated minimal orthogonal main-effect plans (minimal PROMEP's) are OMEF's with two additional properties: (1) Their run numbers reach the lower bounds

of all possible OMEP's with specified factor levels. (2) They are constructed so that a certain number of duplicate points are obtained yet the orthogonality of the plans is kept. The advantages of minimal PROMEP's are obvious. First, this reduces the cost of experiments since the plans have the least run numbers among OMEP's. Second, partial replication facilitates the analysis of experimental data as indicated by Pigeon and McAllister (1989). The pure error can be estimated without assuming the model.

To check the existence of Property (1), Jacroux (1992) provides the sufficient conditions for an OMEP having the desired number of factors and specified levels with a minimal number of runs. Theorem 1 reproduces the sufficient conditions.

Theorem 1 (Jacroux, 1992) *Suppose that an OMEP d has $k \geq 3$ factors in which factor i has s_i levels, $i = 1, \dots, k$, $s_1 \geq s_2 \geq \dots \geq s_k$ and N runs. If $N = \tilde{s}_1 \tilde{s}_2$ for $\tilde{s}_1 \geq s_1$ and $\tilde{s}_2 \geq s_2$ satisfying*

$$\tilde{s}_1 \tilde{s}_2 = \min_{x \geq s_1, y \geq s_2} xy < 2s_1 s_2, \quad s_3 \{\text{lcm}(x, y)\} \leq xy$$

where $\text{lcm}(x, y)$ denotes the least common multiple between positive integers x and y , then d is a minimal OMEP.

Property (2) can be obtained through processes such as the interchange algorithm, special arrangements of the plans, and the utilization of an appropriate collapsing scheme (e.g., see Burgess and Street (1994), Burgess and Street (1999), Chang (1998), Jacroux (1992, 1993), Street (1994)). The appropriate collapsing scheme is a simple and rapid means of ensuring a certain number of duplicate points in the plan. Herein, "collapsing" is a common technique used during the construction of OMEP's. Assume that $m_1 > m_2$, an m_1 -level factor can be collapsed to an m_2 -level factor by applying many-to-one mapping from the set of m_1 levels into the set of m_2 levels. Addelman (1962) verifies that the

orthogonality of the plan is maintained during collapsing. Under the usage of proper collapsing schemes, Chang (2003) further develops a generalization of the construction of Jacroux (1992). In particular, the restriction of \tilde{s}_1 being a multiple of \tilde{s}_2 is removed. This is the construction method adopted by MinPROMEP. In next section, the method is introduced. In Section 3, the program algorithm is described. Finally, the using of MinPROMEP is demonstrated and illustrative examples are given.

2 The Construction Methods in MinPROMEP

Two major theorems used in MinPROMEP are restated as the follows (Chang, 2003).

Theorem 2 (Chang, 2003) *Assume that $\tilde{s}_1 \geq \tilde{s}_2$ satisfies the sufficient conditions of Theorem 1 and \tilde{s}_3 equals $\gcd(\tilde{s}_1, \tilde{s}_2)$. Assume further that q orthogonal latin squares of size $\tilde{s}_3 \times \tilde{s}_3$ exist. An $\tilde{s}_1 \times \tilde{s}_2 \times t_1 \times \cdots \times t_q // \tilde{s}_1 \tilde{s}_2$ minimal OMEP D can be constructed for $q+2$ factors where $t_1 = \cdots = t_q = \tilde{s}_3$.*

Theorem 3 (Chang, 2003) *Assume that $\tilde{s}_1 \geq \tilde{s}_2$ satisfies the sufficient conditions of Theorem 1 and \tilde{s}_3 equals $\gcd(\tilde{s}_1, \tilde{s}_2)$. Assume also that there exists q orthogonal latin squares of size $\tilde{s}_3 \times \tilde{s}_3$. A minimal OMEP d with partial replication can be constructed for $q+2$ factors with factor level number $s_1 \geq s_2 \geq \cdots \geq s_{q+2}$, $s_1 \leq \tilde{s}_1$, $s_2 \leq \tilde{s}_2$ and $s_3 \leq \tilde{s}_3$.*

Class 1: *If $\tilde{s}_1 > s_1$, $\tilde{s}_2 = s_2$, a minimal OMEP with $(\tilde{s}_1 - s_1) \times \tilde{s}_2$ duplicate points can be obtained by appropriately collapsing \tilde{s}_1 levels of factor 1 to s_1 levels;*

Class 2: *If $\tilde{s}_1 = s_1$, $\tilde{s}_2 > s_2$, a minimal OMEP with a certain number of duplicate points can be obtained by appropriately collapsing the \tilde{s}_2 levels of factor 2 to s_2 levels;*

$\tilde{s}_1 \times (\tilde{s}_2 - s_2)$ duplicate points can be obtained if $s_2 > \tilde{s}_3$ and $(\tilde{s}_1 / \tilde{s}_2) \times (\tilde{s}_2 - s_2)$ duplicate points can be obtained if $s_2 \leq \tilde{s}_3$;

Class 3: If $\tilde{s}_1 > s_1, \tilde{s}_2 > s_2$, a minimal OMEP with a certain number of duplicate points can be obtained by appropriately collapsing \tilde{s}_1 levels of factor 1 to s_1 levels, and \tilde{s}_2 levels of factor 2 to s_2 levels: $(\tilde{s}_1 - s_1) \times s_2 + \tilde{s}_1 \times (\tilde{s}_2 - s_2)$ duplicate points can be obtained if $s_2 > \tilde{s}_3$ and $(\tilde{s}_1 / \tilde{s}_2) \times (\tilde{s}_2 - s_2) + (\tilde{s}_1 - s_1) \times \tilde{s}_2$ duplicate points can be obtained if $s_2 \leq \tilde{s}_3$;

Class 4: If $\tilde{s}_1 = s_1, \tilde{s}_2 = s_2$, a minimal OMEP cannot have any duplicate points.

In fact, Theorem 3 ensures that maximum number of duplicate points are obtained when (1) $\tilde{s}_1 > s_1$, and $\tilde{s}_2 = s_2$, (2) $\tilde{s}_1 = s_1$ and $\tilde{s}_2 > s_2 > \tilde{s}_3$, and (3) $\tilde{s}_1 > s_1$ and $\tilde{s}_2 > s_2 > \tilde{s}_3$.

The algorithm developed here also considers the case of three-factor minimal PROMEP's for factor levels $s_1 \times (s_1 - 1)^2$ when $s_1 \geq 5$. A related corollary is reproduced in the following. The constructed minimal PROMEP contains maximum number of duplicate points.

Corollary 1 (Chang, 1998) *A three-factor OMEP having levels of factors $s_1 \times (s_1 - 1)^2$ exists with minimal runs and maximum duplicate points for $s_1 \geq 5$.*

3 Description of the Proposed Algorithm

The largest three factor levels – s_1, s_2 and s_3 as well as the name of the dataset storing the plan are the required input. By receiving the specified factor levels – s_1 and s_2 , \tilde{s}_1 and \tilde{s}_2 are then calculated according to Theorem 1. Set \tilde{s}_3 as the greatest common divisor of \tilde{s}_1 and \tilde{s}_2 . Next, a minimal OMEP with $\tilde{s}_3 + 1$ factors is generated if \tilde{s}_3 is

a prime or a power of a prime. Otherwise, a three-factor minimal PROMEP is generated.

The output dataset contains a minimal PROMEP, $s_1 \times s_2 \times s_3^{k-2} // \tilde{s}_1 \tilde{s}_2$, with a certain number of duplicate points. Herein, $s_1 \times s_2 \times s_3^{k-2} // \tilde{s}_1 \tilde{s}_2$ represents an OMEP having one s_1 -level factor, one s_2 -level factor and $k-2$ factors of s_3 level in $\tilde{s}_1 \tilde{s}_2$ runs. Moreover, when $s_2 = s_3 = s_1 - 1$ and $s_1 \geq 5$, a three-factor minimal PROMEP with maximum number of duplicate points is also generated according to Corollary 1.

In this program, two constraints are imposed for the parameters: (a) $s_3 \geq 2$, and (b) $\tilde{s}_3 \leq 20$.

The following describes the algorithm in steps.

Step 1 When $s_2 = s_3 = s_1 - 1$ and $s_1 \geq 5$, a three-factor minimal PROMEP with maximum duplicate points is generated according to Corollary 1. Print out the plan.

Step 2 Calculate \tilde{s}_1 and \tilde{s}_2 according to Theorem 1. Calculate \tilde{s}_3 .

Step 3 If \tilde{s}_3 is a prime or a power of a prime, generate the minimal OMEP d_0 in $\tilde{s}_3 + 1$ factors; otherwise, generate the minimal OMEP d_0 in three factors.

Step 4 Collapse factor 1 into the s_1 level, and factor 2 into the s_2 level as described in the proof of Theorem 3 for design d_0 . Collapse the rest of the factors into s_3 level.

Step 5 Print out the generated minimal PROMEP.

4 The operation of MinPROMEP

Assume that file “all.sas” contains all macros and is inserted from disk driver “A:\”. Since major macro is named “MinPROMEP.sas” in this study, type in the commands in the SAS environment as follows.

%include “A:\all.sas”;

%MinPROMEP(s₁, s₂, s₃, dsn);

Here, s_1 , s_2 and s_3 are the first three largest factor levels and dsn is the desired dataset name for output. The output includes the generated plan and the number of duplicated points produced. If the maximum number of duplicate points for the plan is reached, a confirmation remark is given in the footnote.

5 Illustrative Examples

In the following, five examples are given for various classes of plans in Theorem 3. Moreover, Example 1 constructs the design by Corollary 1. Example 5 demonstrates the construction when \tilde{s}_3 is neither a prime nor a power of a prime.

Example 1: Submit the following commands

%include "A:\all.sas";

%MinPROMEP(5, 4, 4, dsn1);

Then, two plans are generated and stored in two datasets. One is the $5 \times 4^2 // 25$ minimal PROMEP with maximum number of duplicate points by Corollary 1. The plan is stored in the file named by an extra "A" in front of user specified dataset name. The other is the $5 \times 4^5 // 49$ minimal PROMEP with one duplicate point by Theorem 3. See Output Listing 1(a) and 1(b) for the two plans.

Output Listing 1(a):

The 5 x 4 x 4 // 5 * 5 minimal PROMEP
with 2 duplicate points
(Dataset Name: Adsn1)

Obs	F1	F2	F3
1	0	0	0
2	0	0	0
3	0	1	2
4	0	2	1

5	0	3	3
6	1	0	0
7	1	0	0
8	1	1	3
9	1	2	2
10	1	3	1
11	2	0	1
12	2	0	2
13	2	1	0
14	2	2	3
15	2	3	0
16	3	0	3
17	3	0	1
18	3	1	0
19	3	2	0
20	3	3	2
21	4	0	2
22	4	0	3
23	4	1	1
24	4	2	0
25	4	3	0

* Note: This is a minimal PROMEP with maximum duplicate points.

Output Listing 1(b):

The 5 x 4 x 4 ^4 // 5 * 5 minimal PROMEP
with 1 duplicate points
(Dataset Name: dsn1)

Obs	F1	F2	F3	F4	F5	F6
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	1	1	1	1	1
4	0	2	2	2	2	2
5	0	3	3	3	3	3
6	1	0	0	1	2	3
7	1	0	1	2	3	0
8	1	1	2	3	0	0
9	1	2	3	0	0	1
10	1	3	0	0	1	2
11	2	0	1	3	0	2
12	2	0	2	0	1	3
13	2	1	3	0	2	0
14	2	2	0	1	3	0
15	2	3	0	2	0	1
16	3	0	2	0	3	1
17	3	0	3	1	0	2
18	3	1	0	2	0	3
19	3	2	0	3	1	0
20	3	3	1	0	2	0
21	4	0	0	3	2	1
22	4	0	3	2	1	0

23	4	1	0	0	3	2
24	4	2	1	0	0	3
25	4	3	2	1	0	0

Example 2: Submit the following commands

```
%include "A:\all.sas";
```

```
%MinPROMEP(7, 4, 4, dsn2);
```

Since $\tilde{s}_1 = 8, \tilde{s}_2 = 4$, this is a case of Class 1 in Theorem 3 that $\tilde{s}_1 > s_1, \tilde{s}_2 = s_2$. The $7 \times 4^4 // 32$ minimal PROMEP with 4 duplicate points is generated. The plan obtains maximum number of duplicate points. See Output Listing 2 for the plan.

Output Listing 2:

The 7 x 4 x 4 ^3 // 8 * 4 minimal PROMEP
with 4 duplicate points
(Dataset Name: dsn2)

Obs	F1	F2	F3	F4	F5
1	0	0	0	0	0
2	0	1	1	1	1
3	0	2	2	2	2
4	0	3	3	3	3
5	1	0	1	2	3
6	1	1	0	3	2
7	1	2	3	0	1
8	1	3	2	1	0
9	2	0	2	3	1
10	2	1	3	2	0
11	2	2	0	1	3
12	2	3	1	0	2
13	3	0	3	1	2
14	3	0	3	1	2
15	3	1	2	0	3
16	3	1	2	0	3
17	3	2	1	3	0
18	3	2	1	3	0
19	3	3	0	2	1
20	3	3	0	2	1
21	4	0	0	0	0
22	4	1	1	1	1
23	4	2	2	2	2
24	4	3	3	3	3
25	5	0	1	2	3
26	5	1	0	3	2
27	5	2	3	0	1
28	5	3	2	1	0

29	6	0	2	3	1
30	6	1	3	2	0
31	6	2	0	1	3
32	6	3	1	0	2

* Note: This is a minimal PROMEP with maximum duplicate points.

Example 3: Submit the following commands

```
%include "A:\all.sas";
```

```
%MinPROMEP(4, 3, 3, dsn3);
```

Since $\tilde{s}_1 = 4, \tilde{s}_2 = 4$, this is a case of Class 2 in Theorem 3 that $\tilde{s}_1 = s_1, \tilde{s}_2 > s_2$ and $s_2 < \tilde{s}_3$. The plan generates $4 \times 3 \times 3^3 // 16$ minimal PROMEP with 1 duplicate point. It can be verified that the maximal number of duplicate points for plan $4 \times 3^4 // 16$ equals 1. See Output Listing 3 for the plan.

Output Listing 3:

The $4 \times 3 \times 3^3 // 4 \times 4$ minimal PROMEP
with 1 duplicate points
(Dataset Name: dsn3)

Obs	F1	F2	F3	F4	F5
1	0	0	0	0	0
2	0	0	0	0	0
3	0	1	1	1	1
4	0	2	2	2	2
5	1	0	1	2	0
6	1	0	2	1	0
7	1	1	0	0	2
8	1	2	0	0	1
9	2	0	1	0	2
10	2	0	2	0	1
11	2	1	0	2	0
12	2	2	0	1	0
13	3	0	0	1	2
14	3	0	0	2	1
15	3	1	2	0	0
16	3	2	1	0	0

Example 4: Submit the following commands

```
%include "A:\all.sas";
```

`%MinPROMEP(11, 7, 3, dsn4);`

Since $\tilde{s}_1 = 12, \tilde{s}_2 = 8$, this is a case of Class 3 in Theorem 3 that $\tilde{s}_1 > s_1, \tilde{s}_2 > s_2$ and $s_2 > \tilde{s}_3$. The plan generates $11 \times 7 \times 3^3 // 96$ minimal PROMEP with the maximum of 19 duplicate points. See Output Listing 4 for the plan.

Output Listing 4:

The $11 \times 7 \times 3^3 // 12 * 8$ minimal PROMEP
with 19 duplicate points
(Dataset Name: dsn4)

Obs	F1	F2	F3	F4	F5
1	0	0	0	0	0
2	0	1	1	1	1
3	0	2	2	2	2
4	0	3	0	0	0
5	0	3	0	0	0
6	0	4	0	0	0
7	0	5	1	1	1
8	0	6	2	2	2
9	1	0	1	2	0
10	1	1	0	0	2
11	1	2	0	0	1
12	1	3	2	1	0
13	1	3	2	1	0
14	1	4	1	2	0
15	1	5	0	0	2
16	1	6	0	0	1
17	2	0	2	0	1
18	2	1	0	2	0
19	2	2	0	1	0
20	2	3	1	0	2
21	2	3	1	0	2
22	2	4	2	0	1
23	2	5	0	2	0
24	2	6	0	1	0
25	3	0	0	1	2
26	3	0	0	1	2
27	3	1	2	0	0
28	3	1	2	0	0
29	3	2	1	0	0
30	3	2	1	0	0
31	3	3	0	2	1
32	3	3	0	2	1
33	3	3	0	2	1
34	3	3	0	2	1
35	3	4	0	1	2

36	3	4	0	1	2
37	3	5	2	0	0
38	3	5	2	0	0
39	3	6	1	0	0
40	3	6	1	0	0
41	4	0	0	0	0
42	4	1	1	1	1
43	4	2	2	2	2
44	4	3	0	0	0
45	4	3	0	0	0
46	4	4	0	0	0
47	4	5	1	1	1
48	4	6	2	2	2
49	5	0	1	2	0
50	5	1	0	0	2
51	5	2	0	0	1
52	5	3	2	1	0
53	5	3	2	1	0
54	5	4	1	2	0
55	5	5	0	0	2
56	5	6	0	0	1
57	6	0	2	0	1
58	6	1	0	2	0
59	6	2	0	1	0
60	6	3	1	0	2
61	6	3	1	0	2
62	6	4	2	0	1
63	6	5	0	2	0
64	6	6	0	1	0
65	7	0	0	1	2
66	7	1	2	0	0
67	7	2	1	0	0
68	7	3	0	2	1
69	7	3	0	2	1
70	7	4	0	1	2
71	7	5	2	0	0
72	7	6	1	0	0
73	8	0	0	0	0
74	8	1	1	1	1
75	8	2	2	2	2
76	8	3	0	0	0
77	8	3	0	0	0
78	8	4	0	0	0
79	8	5	1	1	1
80	8	6	2	2	2
81	9	0	1	2	0
82	9	1	0	0	2
83	9	2	0	0	1
84	9	3	2	1	0
85	9	3	2	1	0
86	9	4	1	2	0
87	9	5	0	0	2
88	9	6	0	0	1
89	10	0	2	0	1

90	10	1	0	2	0
91	10	2	0	1	0
92	10	3	1	0	2
93	10	3	1	0	2
94	10	4	2	0	1
95	10	5	0	2	0
96	10	6	0	1	0

* Note: This is a minimal PROMEP with maximum duplicate points.

Example 5: Submit the following commands

```
%include "A:\all.sas";
```

```
%MinPROMEP(11, 6, 6, dsn5);
```

Since $\tilde{s}_1 = 12, \tilde{s}_2 = 6$, this is a case of Class 1 in Theorem 3. The plan generates $11 \times 6^2 // 72$ minimal PROMEP with 6 duplicate points. The plan obtains maximum number of duplicate points. See Output Listing 5 for the plan.

Output Listing 5:

The 11 x 6 x 6 // 12 * 6 minimal PROMEP
with 6 duplicate points
(Dataset Name: dsn5)

Obs	F1	F2	F3
1	0	0	0
2	0	1	1
3	0	2	2
4	0	3	3
5	0	4	4
6	0	5	5
7	1	0	1
8	1	1	2
9	1	2	3
10	1	3	4
11	1	4	5
12	1	5	0
13	2	0	2
14	2	1	3
15	2	2	4
16	2	3	5
17	2	4	0
18	2	5	1
19	3	0	3
20	3	1	4
21	3	2	5
22	3	3	0
23	3	4	1
24	3	5	2

25	4	0	4
26	4	1	5
27	4	2	0
28	4	3	1
29	4	4	2
30	4	5	3
31	5	0	5
32	5	0	5
33	5	1	0
34	5	1	0
35	5	2	1
36	5	2	1
37	5	3	2
38	5	3	2
39	5	4	3
40	5	4	3
41	5	5	4
42	5	5	4
43	6	0	0
44	6	1	1
45	6	2	2
46	6	3	3
47	6	4	4
48	6	5	5
49	7	0	1
50	7	1	2
51	7	2	3
52	7	3	4
53	7	4	5
54	7	5	0
55	8	0	2
56	8	1	3
57	8	2	4
58	8	3	5
59	8	4	0
60	8	5	1
61	9	0	3
62	9	1	4
63	9	2	5
64	9	3	0
65	9	4	1
66	9	5	2
67	10	0	4
68	10	1	5
69	10	2	0
70	10	3	1
71	10	4	2
72	10	5	3

* Note: This is a minimal PROMEP with maximum duplicate points.

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