



## **BiDimRegression: Bidimensional Regression Modeling Using R**

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### **Abstract**

Tobler (1965) introduced bidimensional regression to the research field of geography in 1965 to provide a method for estimating mapping relations between two planes on the basis of regression modeling. The bidimensional regression method has been widely used within geographical research. However, the applicability in assessing the degree of similarity of two-dimensional patterns has not much explored in the area of psychological research, particularly in the domains of cognitive maps, face research and comparison of 2D-data patterns. Describing Tobler's method in detail, Friedman and Kohler (2003) made an attempt to bridge the gulf between geographical methodological knowledge and psychological research practice. Still, the method has not been incorporated into psychologists' standard methodical repertoire to date. The present paper aims to make bidimensional regression applicable also for researchers and users unfamiliar with its theoretical basis. The `BiDimRegression` function provides a manageable computing option for bidimensional regression models with affine and Euclidean transformation, which makes it easy to assess the similarity of any planar configuration of points. Typical applications are, for instance, assessments of the similarity of facial images defined by discrete features or of (cognitive) maps characterized by landmarks. `BiDimRegression` can be a valuable tool since it provides estimation, statistical inference, and goodness-of-fit measures for bidimensional regression.

*Keywords:* bidimensional regression, R, calculation, probability, similarity, Euclidean, affine, projective, nonlinear, inference statistics, psychology, geography, cognitive cartography, open source.

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## **1. Motivation**

This paper aims to make an important methodological procedure more manageable and, thus, also popular. Waldo R. Tobler (1965), a geographer and cartographer, was the first

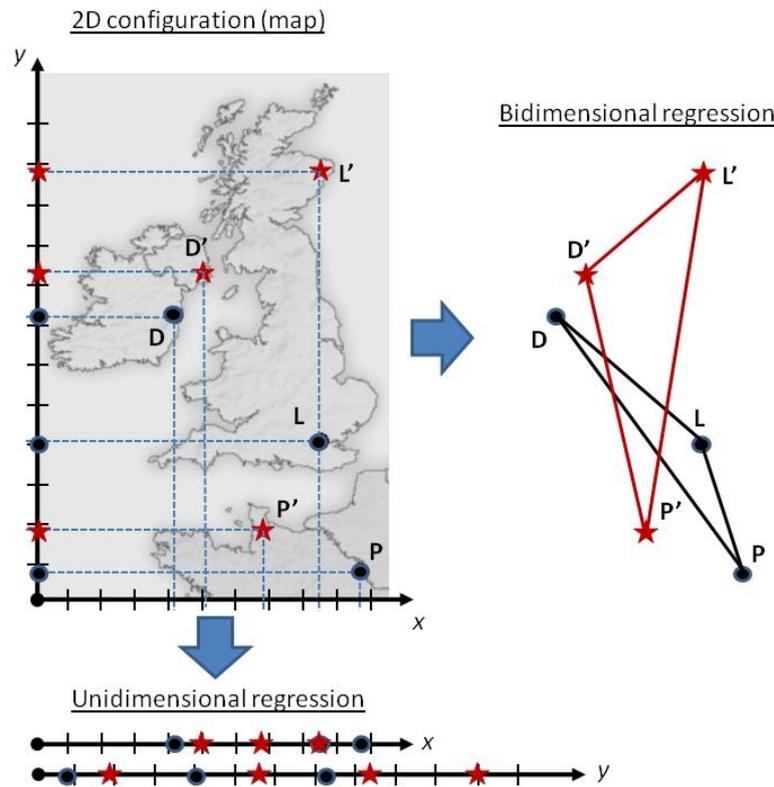


Figure 1: Comparison of unidimensional and bidimensional regression.

to introduce the method of bidimensional regression to the scientific audience of geography. His main aim was to estimate the degree of correspondence between two plane patterns of point locations as typically shown in geographical maps. The essential difference between bidimensional regression and standard regression procedures is that the data input consists of paired locations or “bidimensional” coordinates (e.g.,  $[x, y; x', y']$ ) instead of unpaired “unidimensional” coordinates (e.g.,  $[x; x']$  and  $[y; y']$ ). With one word, for bidimensional regressions, the relationship between two variables is assessed by referring to measures on two conjoint dimensions, for instance the 2D-coordinates of a point on a plane, whereas for unidimensional regressions the different dimensions of coordinates are analyzed independently from each other. In the example of map comparison (Figure 1), this difference can easily be demonstrated. Whereas the projection of two separate dimensions realized by the approach of unidimensional regression loses important information on the 2D-coordinates, the bidimensional approach still uses the full 2D-coordinate information. In the given example, the clear dislocations of all cities in terms of relations to each other could not be visible on the analysis of single dimensions, not linked together. More generally, any kind of complex 2D-dislocations, which do not follow a single distortion factor on one single dimension, could not be described and interpreted in an adequate way if only unidimensional regressions are used. Figure 1 illustrates the comparison of the commonly used unidimensional regression and the bidimensional regression with European cities. Real positions of the cities (D)ublin, (L)ondon and (P)aris are shown as solid black circles and fictive positions of corresponding cities (D', L' and P'), whose positions should be compared with the original locations, are shown as

red stars. The unidimensional regression splits all 2D coordinates from the original map into two separate, unlinked vectors, thus, being only capable of assessing similarity between both configurations on single dimensions. The bidimensional regression, instead, uses the full 2D configurations of both position sets to assess the similarity.

The bidimensional regression procedure introduced by Tobler mainly returns a similarity measure between two 2D configurations via a regression coefficient compatible with Pearson's  $R$ . The squared bidimensional regression coefficient  $R^2$  can concordantly be interpreted as the amount of explained variance. The function provides information on the translation, scaling, and rotation and whether the source configuration has to be mirrored to produce the best fit for the target configuration. As the result of the bidimensional regressions contains already the full pack of these information no explicit procrustes transformation has to be conducted. Most importantly for scientific usage, the here presented R function calculates additional information on the statistical significance of the targeted regression model as well as the comparative model of different geometry based on  $F$  statistics and outputs the regression coefficients along with their significance levels based on  $t$  statistics described by Nakaya (1997).

## 2. Preposition

Friedman and Kohler (2003) made the first important step to introducing Tobler's bidimensional regression model to a non-geographical audience, not least by providing, in the form of a Microsoft **Excel** sheet, the first openly accessible tool for calculating bidimensional regressions based on Euclidean geometry. By means of the present R function Friedman and Kohler's approach shall be extended expanding the capabilities of the implemented method with regard to several aspects: (1) affine geometry is integrated, (2) inferential statistics on the models and the respective parameters are calculated, and (3) the full source code is provided in R, which enables researchers to automatize bidimensional regression procedures and, thus, integrate them into everyday scientific work. Further details on the mathematical basis of the procedure can be found in the original papers of Tobler (e.g., 1965, 1966, 1978, 1994). Besides, I would like to refer to Nakaya (1997) for the stochastic basis of  $F$  and  $t$  statistics applied to bidimensional regression, and, last but not least, to Friedman and Kohler (2003) for a step-by-step description of the logic of bidimensional regression. Instead of merely reiterating what has already been stated, the present paper, thus, aims at integrating these different contributions within a straightly applicable, easy-to-use R function that should facilitate an incorporation of bidimensional regression into the standard repertoire of psychological research. This intention is very much in accordance with Tobler's forward-looking view on the potential applicability of bidimensional regression to sciences outside the field of geography: "The method seems particularly useful for geographers [...] but does not appear to have been applied in social sciences" (Tobler 1965, pp. 131).

## 3. Implementation

The **BiDimRegression** package has been implemented and tested in ordinary R (R 2.15.2, R Core Team 2012). It is available from the Comprehensive R Archive Network at <http://CRAN.R-project.org/package=BiDimRegression> and depends on the **nlme** package (Pinheiro, Bates, DebRoy, Sarkar, and R Core Team 2013). The code is commented throughout

in English. In order to enable researchers to integrate the bidimensional regression procedure into their own calculation functions, the code has been implemented as an R function with simple data input and output. As this R function provides the calculation of bidimensional regressions (i.e., the assessment of similarities between two 2D configurations) based on Euclidean and affine geometries along with the analysis of several helpful relevant parameters and the corresponding inference statistics, it seems to be particularly helpful and applicable for any statistically oriented research area.

The call of the R function `BiDimRegression` is:

```
R> resultingMeasures <- BiDimRegression(coord)
```

**Input parameter:** `coord` stands for the R `data.frame` containing the coordinates of the independent as well as the dependent configurations. According to the notation system of Friedman and Kohler (2003) this R `data.frame` contains vectors with the following names:

1. Vectors `A`, `B` provide the coordinates of the dependent configuration, i.e., the configuration which emerges from the independent (original) one, for instance, a cognitive map. `A` refers to the first and `B` to the second dimension.
2. Vectors `X`, `Y` provide the coordinates of the independent configuration, i.e., the original or base configuration, for instance, a geographical map. `X` refers to the first and `Y` to the second dimension.

Several examples can be loaded as R `data.frame` via `data(dataset, package = "BiDimRegression")`, where `dataset` refers to the considered data, e.g., "`FriedmanKohlerData1`" for example 1 by Friedman and Kohler (2003). The available datasets are:

- "`NakayaData`": contains the example from Nakaya (1997, Table 2).
- "`FriedmanKohlerData1`": "Case 1" from Friedman and Kohler (2003).
- "`FriedmanKohlerData2`": "Case 2" from Friedman and Kohler (2003).
- "`CarbonExample1Data`": Example 1 on "Mona Lisa's two faces" from the present article (see Section 6.1).
- "`CarbonExample2Data`": Example 2 on "Maps of Paris" from the present article (see Section 6.2).
- "`CarbonExample3Data`": Example 3 on "Cognitive vs. geographic maps" from the present article (see Section 6.3).

**Output values:** Statistics on the overall fit of the bidimensional regressions for Euclidean and affine geometries as well as detailed further information and statistics, organized as a table (left column: Euclidean geometry, right column: affine geometry):

1. *Date/time/processing time*: Each output starts with a protocol of the used function including a date/time stamp plus the version of the employed R function. For reasons of facility of inspection in case of repeated runs of the function, the processing time needed for each calculation is displayed in seconds.

2.  $r$ ,  $r^2$ : Regression coefficient ( $R$ ) and the squared regression coefficient ( $R^2$ ), respectively.
3.  $F$  statistics:  $F$  statistics for the overall regression model including appendant degrees of freedom ( $df1$ ,  $df2$ ),  $p$  value and, as for all following statistics as well, the accordant significance level ( $*** \hat{=} p < 0.001$ ,  $** \hat{=} p < 0.01$ ,  $* \hat{=} p < 0.05$ , and  $n.s. \hat{=} non\ significant$ ). As proposed by Nakaya (1997), significance tests for the bidimensional regression are executed on basis of  $F$  distributions with  $df1 = p - 2$  (degrees of freedom of the nominator) and  $df2 = 2n - p$  (degrees of freedom of the denominator). According to Nakaya (1997, Table 1) parameter  $p$  is defined as the number of elements needed to calculate the associated model with  $p = 4$  for Euclidean and  $p = 6$  for affine geometry; parameter  $n$  is defined as the number of coordinate pairs.
4.  $DAIC$  statistics: Following formulas (52–56) in Nakaya (1997), the  $AIC$  (Akaike Information Criterion) is calculated for the referring bidimensional model against the bidimensional null model. The  $AIC$  is an indicator that accounts for "the principle of parsimony" following the idea that a superior model is one that is not only characterized by a higher goodness-of-fit, but also by being more frugal. If  $DAIC$  (i.e., the difference of both  $AIC$ s) is smaller than zero, the bidimensional regression model is better than the bidimensional null model.
5.  $alpha1$ – $alpha2$ : Following the notation of Friedman and Kohler (2003), the  $alphas$  reflect the intercept vectors of the bidimensional regression equation shown below.
6.  $beta1$ – $beta4$ : Following the notation of Friedman and Kohler (2003), the  $betas$  reflect the slope vectors of the bidimensional regression equation shown below;  $beta3$  and  $beta4$  are not available for Euclidean geometry.
7.  $scaleX$ : Scaling factor on the first dimension (1.0 means no scaling; values below 1.0 indicate a contraction; values above 1.0 indicate an expansion).
8.  $scaleY$ : Scaling factor on the second dimension. Per definition,  $scaleY$  equals  $scaleX$  for the Euclidean geometry.
9.  $angle$ : Rotation angle (in degrees).
10.  $shear$ : Information on the shearing of the transformed configuration, see Friedman and Kohler (2003, p. 490). Per definition,  $shear$  always equals zero for Euclidean geometry.
11.  $Distortion\ distance$  statistics: Following the extension of the bidimensional regression developed by Waterman and Gordon (1984) and refined by Friedman and Kohler (2003),  $distortion\ distance$  statistics calculate the distortion of a dependent configuration from the original one. In line with Friedman and Kohler (2003)  $diABSqr$  and  $diXYSqr$  (with "Sqr" always indicating squared values) are displayed as proportions of unexplained variance. In contrast to an earlier suggestion by Waterman and Gordon (1984) these values are not multiplied by 100, so values range from 0 to 1 (see for details Friedman and Kohler 2003, Footnote 1).
12.  $Comparative\ test$  statistics: To allow comparisons between the Euclidean and the affine solution, two different comparative test statistics are provided: (1) An  $F$  statistic based on formula (58) in Nakaya (1997) taking sum of squares and the number of parameters of both models into account and (2) an approach based on

AIC statistics, described by formula (62) in Nakaya (1997) calculating AIC, the difference of both AIC values. Statistical significance is indicated by the  $F$  statistic. In the case of significance ( $p < 0.05$ ), negative values for  $DAIC$  are interpreted as indicating a superior affine solution, while positive values indicate a superior Euclidean solution.

Note: The main measures are additionally returned from the function to the `data.frame` `resultingMeasures` consisting of two further `data.frame` sub-structures labeled *euclidean* and *affine*.

The main equation for the Euclidean bidimensional regression with the independent coordinates  $A$  and  $B$  and the dependent coordinates  $X$  and  $Y$  is given by

$$(X^\top Y^\top) = \alpha + \beta \cdot (AB) \quad (1)$$

which transforms to the following matrix version for Euclidean geometry

$$\begin{pmatrix} X^\top \\ Y^\top \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} \quad (2)$$

Note that like in the case of unidimensional regression it is only a matter of definition which coordinates are labelled as independent and dependent, respectively. Analogous to unidimensional regression it is, though, helpful to theorize which configuration is to be taken as the given (the ‘‘original’’) one and which configuration has evolved from it. In the case of cognitive maps, this decision can easily be made: It seems to be most useful and sensible here to define a present (geographical) map as source configuration or independent variable set, and to define the cognitive map that might, for instance, have been produced using multidimensional scaling of distance estimations as dependent variable set. In the case of comparing, for instance, two historic maps or any other 2D-data without any concept of originality, this clear-cut dissociation between independent and dependent variable set is no longer possible. Furthermore, it is quite essential to interpret the returning values and calculations in direction from the independent towards the dependent variable set, e.g., a return value of  $angle = 45$  would mean that the ‘‘independent’’ configuration would optimally fit the ‘‘dependent’’ configuration by clockwise rotating the independent configuration by  $45^\circ (= \pi/4)$ .

#### 4. Statistical testing of the models and the parameters

Importantly, Nakaya (1997) extended Tobler’s seminal idea of bidimensional regression by implementing a framework of inference statistics: In formulas (50) and (33) of his original article he introduced  $F$  statistics for testing the overall estimated model and  $t$  tests for testing the significance of the respective model parameters. While adopting the  $F$  statistics for overall model testing the present R function implemented a much simpler approach for testing the model’s parameters  $\{\alpha, \alpha_2\}$  and  $\{\beta_1, \beta_2, \beta_3, \beta_4\}$  against zero utilizing the standard R function `lm` for testing linear models. To use this standard R function, an ordinary unidimensional multiple regression without intercept has to be conducted on basis of *design matrices* of the referring geometries ( $\mathbf{D}_e$  for Euclidean and  $\mathbf{D}_a$  for affine geometry) as described in Nakaya (1997, Table 1):

$$D_e = \begin{pmatrix} 1 & 0 & x & -y \\ 0 & 1 & y & x \end{pmatrix} \quad (3)$$

$$D_a = \begin{pmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & y \end{pmatrix} \quad (4)$$

The application of the respective *design matrix* to the parameter vector  $\gamma$ , see Nakaya (1997, Table 1 and Formula 29), yields the following multiple regression model for Euclidean geometry (with  $n$  being the number of coordinate pairs):

$$\omega = D_e^n \gamma + \xi \quad (5)$$

where  $\omega$  is a target vector (dependent variable),  $(u_1, \dots, u_n, v_1, \dots, v_n)^\top$ ,  $\xi$  is a residual vector, and the Euclidean design matrix for  $n$  sample points is written

$$D_e^n = \begin{pmatrix} 1 & 0 & x_1 & -y_1 \\ 1 & 0 & x_2 & -y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_n & -y_n \\ 0 & 1 & y_1 & x_1 \\ 0 & 1 & y_2 & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & y_n & x_n \end{pmatrix} \quad (6)$$

The respective multiple regression model for affine geometry is given by:

$$\omega = D_a^n \gamma + \xi \quad (7)$$

where the affine design matrix for  $n$  sample points is written

$$D_a^n = \begin{pmatrix} 1 & 0 & x_1 & y_1 & 0 & 0 \\ 1 & 0 & x_2 & y_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_n & y_n & 0 & 0 \\ 0 & 1 & 0 & 0 & x_1 & y_1 \\ 0 & 1 & 0 & 0 & x_2 & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & x_n & y_n \end{pmatrix} \quad (8)$$

Comparative testing of the models for Euclidean against affine geometry mainly follows two separate methodological approaches: a) classical theory of inferential statistics and b) calculation of AICs. Further information on both methods can be found in formulas (59) and (62), respectively, explained in detail by Nakaya (1997).

## 5. Application

To demonstrate the functionality and applicability of the `BiDimRegression` R function, we will feed the function with the original data set provided by Friedman and Kohler (2003,

Table 3). As these authors gave an excellent introduction and overview on bidimensional regression, the reader might benefit most by carefully reading their theoretical description of the method when applying the R function presented here. The original dataset can easily be implemented by calling the following command that initializes the vectors of the independent coordinates with  $A = (12, 19, 21, 18)$  and  $B = (16, 19, 18, 14)$  and the vectors of the dependent coordinates with  $X = (0, 10, 8, 6)$  and  $Y = (0, 0, 5, 5)$ :

```
R> data("FriedmanKohlerData1", package = "BiDimRegression")
```

The bidimensional regression will then be executed by:

```
R> BiDimRegression(FriedmanKohlerData1)
```

This command will execute the R function `BiDimRegression` encompassing the calculation of two bidimensional regressions based on Euclidean and affine geometry, respectively, as well as accordant comparative statistics on them. The result is in accord with the outcome shown in Table 3 (“case 1”, top section) provided by [Friedman and Kohler \(2003\)](#), with  $X$ ,  $Y$  as independent variables. If, in contrast,  $A$ ,  $B$  should be defined as the independent variables (“case 2”, bottom section), the referring outcome will be calculated analogously. By filling the R `data.frame` of the input parameter with her own 2D-configurations, the user will be able to calculate bidimensional regressions on these data. To show summary statistics the user can execute the `summary()` method, or, for a more detailed output the `print()` method.

Another example should help to replicate and to reanalyze step-by-step the findings described in detail by [Nakaya \(1997\)](#). The following calls will reproduce the case of 19 to be matched coordinate pairs discussed by [Nakaya \(1997, Table 2\)](#):

```
R> data("NakayaData", package = "BiDimRegression")
R> BiDimRegression(NakayaData)
```

## 6. Further examples on bidimensional data

In the following section I want to demonstrate the applicability of the `BiDimRegression` function by referring to three different datasets I have gathered together over the last years. I have selected example 1 from the domain of aesthetics to show how the method can be utilized for assessing the similarity of two portrayed persons, actually the Mona Lisa in the world famous Louvre version and the only recently re-discovered Prado version. Example 2 originates from the area of geography and inspects the accuracy of different maps of the city of Paris which were created over the last 350 years as compared to a recent map. Finally, example 3 focuses on demonstrating how good a cognitive map recalculated from averaged cognitive distance data fits with a related real map. Therefore two samples of East and West Germans, respectively, will be used to get notion on the typical degree of correspondence between cognitive and real maps in terms of bidimensional regression coefficients.

### 6.1. Example 1: Mona Lisa’s two faces

Bidimensional regression is always of high value if two pictures showing discrete landmarks that can unambiguously matched for both versions have to be compared. Only recently, a second version of the Mona Lisa has been re-discovered in Madrid’s Prado museum and has now

been revealed as a real masterpiece in terms of craftsmanship. The Prado version strongly resembles the Louvre version, despite different colorization stemming from age-related yellowing of the varnish of the Louvre version. To calculate the paintings' similarity on an objective basis, I defined 36 clearly visible landmarks in both versions (tip of the nose, nostrils, hairline, etc.) and fed the `BiDimRegression` function with the coordinates of the Louvre version as independent and the coordinates of the Prado version as dependent variables. The data sets can directly be loaded via `data("CarbonExample1Data", package = "BiDimRegression")` and used via `BiDimRegression(CarbonExample1Data)`. The output, e.g., via `print()`, shows that both paintings are featuring nearly identical landmark aspects (e.g., for the Euclidean version:  $R^2 = 0.998$ , with  $F_{2,68} = 20089.2$ ,  $p < 0.0001$ ).

## 6.2. Example 2: Maps of Paris

The development of bidimensional regression was initiated by geographers to assess the accordance as well as to specify deviations between different geographic maps. By feeding the `BiDimRegression` function with coordinates of different maps of Paris we can calculate the accuracy of these maps in comparison with a standard tourist map of today. To get an imagination of the precision of historic maps I measured the positions of several distinctive landmarks of Paris, which were available over a long period of time: (1) Place de la Concorde, (2) Saint-Germain-des-Prés, (3) Louvre, (4) Théâtre de l'Odéon, (5) Les Halles, (6) La Sorbonne, (7) Notre Dame de Paris, (8) Hôtel de Ville, (9) Temple, (10) La Bastille, (11) Hôtel des Invalides, (12) La Madeleine, (13) École Militaire. By comparing the coordinates of these 13 landmarks as defined by a map originated in 1789 and the modern map, I found a very high correlation between both versions (e.g., for the Euclidean version  $R^2 = 0.996$ , with  $F_{2,22} = 2545.4$ ,  $p < 0.0001$ ). Similar fits were revealed for even older maps, though the number of corresponding landmarks was reduced from 13 to 12 for the maps of 1705 and 1728 and to 10 for the map of 1652. The coordinates of the official map of the Paris metro, known for distortions due to its schematic design, deviated most, but still reached a clear and high fit with the accurate actual map (e.g., for the Euclidean version  $R^2 = 0.938$ , with  $F_{2,22} = 167.6$ ,  $p < 0.0001$ ). The data sets can directly be loaded via `data("CarbonExample2Data", package = "BiDimRegression")` and used via `BiDimRegression(CarbonExample2Data)`.

## 6.3. Example 3: Cognitive vs. geographic maps

The last example also refers to the area of maps; it links cognitive psychology with geography. Approximately 15 years after the German reunification I conducted a study on the cognitive maps of adult Germans who were socialized in the western or the eastern part of Germany (the Federal Republic or the former German Democratic Republic, respectively). In that study I was particularly interested in the so-called "mental wall" between East and West Germany as a function of the social attitude towards the German reunification (Carbon and Leder 2005). For reasons of demonstration typical fits that can be obtained for such cognitive maps, I will now focus on the accuracy of the cognitive maps emerging from averaged data from Germans raised in the western vs. eastern part of the country. In the original task 83 participants were asked to estimate distances across Germany; for the sake of the specific hypothesis we selected only those participants who were clearly socialized in one of the two parts of Germany till the reunification of 1990. This criterion reduced the number of participants to  $n = 57$  (East

Germans:  $n = 25$ , West Germans:  $n = 32$ ). We excluded Berlin from the analysis due to its special status as Berlin could be labeled neither as a (former) western nor as a (former) eastern city. The 45 distances between the 10 remaining cities (eastern cities: Cottbus, Erfurt, Leipzig, Magdeburg, Rostock; western cities: Düsseldorf, Hamburg, Hannover, Nürnberg, Stuttgart) were submitted to MATLAB's `mdscale` function (The MathWorks, Inc. 2011) for nonclassical multidimensional scaling (mds; 2-dimensional solution; *SSTRESS* as convergence criterion) for obtaining 2-dimensional coordinates for each city. The datasets for both samples (East and West Germans) can be found within the `CarbonExample1Data` `data.frame`. For both samples, comparably high fits with the real maps, that were also generated via nonclassical mds, were obtained, e.g., for the Euclidean version:  $R^2 = 0.910$  ( $F_{2,16} = 80.4$ ,  $p < 0.0001$ ) for East Germans and  $R^2 = 0.968$  ( $F_{2,16} = 244.0$ ,  $p < 0.0001$ ) for West Germans. The data sets can directly be loaded via `data("CarbonExample3Data", package = "BiDimRegression")` and used via `BiDimRegression(CarbonExample3Data)`.

## 7. Conclusion

Referring to methodical work elaborated by Tobler (1965, 1994) and other authors like, for instance, Nakaya (1997) who has established inference statistics on the employed methods, the present paper provides an R function for facile calculation of bidimensional regressions for Euclidean as well as for affine geometries. Researchers can quite effortlessly combine this function with their own R functions. The calculated models, which are also analyzed via comparative statistics, are outputted via R console and standardized ASCII `.txt` file. I hope the implementation of this R function that provides an economic means of calculating bidimensional regressions, extended by the capability of conducting according inferential statistics, will contribute to increasing the applicability and publicity of this important method for assessing similarities between any 2D configurations in the context of psychological research, and that it will be incorporated into the standard repertoire of psychologists' statistical expertise.

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