# Implementing Modified Burg Algorithms in Multivariate Subset Autoregressive Modeling

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**Abstract**

The large number of parameters in subset vector autoregressive models often leads one to procure fast, simple, and efficient alternatives or precursors to maximum likelihood estimation. We present the solution of the multivariate subset Yule-Walker equations as one such alternative. In recent work, Brockwell, Dahlhaus, and Trindade (2002), show that the Yule-Walker estimators can actually be obtained as a special case of a general recursive Burg-type algorithm. We illustrate the structure of this Algorithm, and discuss its implementation in a high-level programming language. Applications of the Algorithm in univariate and bivariate modeling are showcased in examples. Univariate and bivariate versions of the Algorithm written in Fortran 90 are included in the appendix, and their use illustrated.

*Keywords:* binary tree, Fortran 90, pointer linked list, recursive algorithm, Yule-Walker estimation

## 1 Introduction

AutoRegressive Moving Average (ARMA) models are well-known and popular in the time series literature. Among others, they are extensively used to model economic and electrical systems whose evolution in time (modulo preliminary transformations, de-trending, and de-seasonalizing) can be well approximated by that of a *stationary* process. Stationarity constrains a process to have second order properties that do not evolve with time, i.e.

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constant mean, and a covariance structure between any two observations that depends only on the distance in time (the *lag*) separating them.

In the multivariate setting, one attempts to model the joint behavior of several univariate series over the same span of time. The usefulness of multivariate ARMA models here though, is stymied by identifiability issues concerning the model parameters (see for example Brockwell and Davis 1991, sec. 11.5). A common alternative is to restrict attention to two special cases: invertible Moving Average (MA) models, and causal AutoRegressive (AR) models, whose parameters are uniquely determined by the second order properties of the process. AR models (called *Vector Autoregressive* (VAR) in the multivariate case) are often favored over MA models due to their interpretability, simplicity of estimation, and ease of forecasting. They are extensively used in signal processing for modeling various phenomena associated with speech and audio; see for example Godsill and Rayner (1998).

The $d$-dimensional vector process $\{\mathbf{X}_t,\ t = 0, \pm 1, \ldots\}$, is said to be a VAR process of order $p$, VAR($p$), if it is a stationary solution of the equations,

$$\mathbf{X}_t = \Phi(1)\mathbf{X}_{t-1} + \cdots + \Phi(p)\mathbf{X}_{t-p} + \mathbf{Z}_t,$$

where, $\Phi(1), \ldots, \Phi(p)$, are $(d \times d)$ constant matrices (the *VAR coefficient matrices*), and $\{\mathbf{Z}_t\}$ is a sequence of zero-mean uncorrelated random vectors, each with covariance matrix $\Sigma$. We call the process $\{\mathbf{Z}_t\}$ *white noise*, and write $\{\mathbf{Z}_t\} \sim \mathrm{WN}(\mathbf{0}, \Sigma)$. The *autocovariance function* of $\mathbf{X}_t$ is,

$$E[\mathbf{X}_{t+h}\mathbf{X}_t'] = \Gamma(h), \quad h = 0, \pm 1, \ldots.$$

A VAR($p$) is therefore a linear regression of the current value of the series on its previous $p$ values i.e. an *autoregression*. We say that we are modeling the series on the lagged set $\{1, \ldots, p\}$.

One can generalize this to modeling on a lagged *subset*

$$K = \{k_1, \ldots, k_m\} \subseteq \{1, \ldots, p\}, \quad \text{with } k_1 < \cdots < k_m \equiv p,$$

and the coefficient matrices pertaining to the lags not present in the set $K$, constrained to be zero. Such models are called *Subset Vector AutoRegressive* (SVAR; SAR in the univariate case), and take the form

$$\mathbf{X}_t = \Phi_K(k_1)\mathbf{X}_{t-k_1} + \cdots + \Phi_K(k_m)\mathbf{X}_{t-k_m} + \mathbf{Z}_t, \quad \{\mathbf{Z}_t\} \sim \mathrm{WN}(\mathbf{0}, U_K). \quad (1)$$

SVAR models are appropriate in situations where one does not wish to include all the lags of the complete (full-set) VAR model. Two such instances are:

**Modeling of seasonal time series.** If $B$ denotes the backward shift operator, i.e. $B^k X_t = X_{t-k}$ for any positive integer $k$, then causal SAR models of the form,

$$(1 - \psi B^s)\,(1 - \phi_1 B - \cdots - \phi_p B^p)\,X_t = Z_t,$$

will exhibit approximate cyclical behavior for appropriate values of the coefficients $\psi,\ \phi_1, \ldots, \phi_p$, and orders $s,\ p$, as evidenced by sharp peaks in the spectral density. This suggests that some seasonal time series can effectively be modeled as SAR processes.

Figure 1 shows a realization from the SAR(3),

$$X_t - 0.99 X_{t-3} = Z_t,\ \{Z_t\} \sim \mathrm{WN}(0, 1),$$

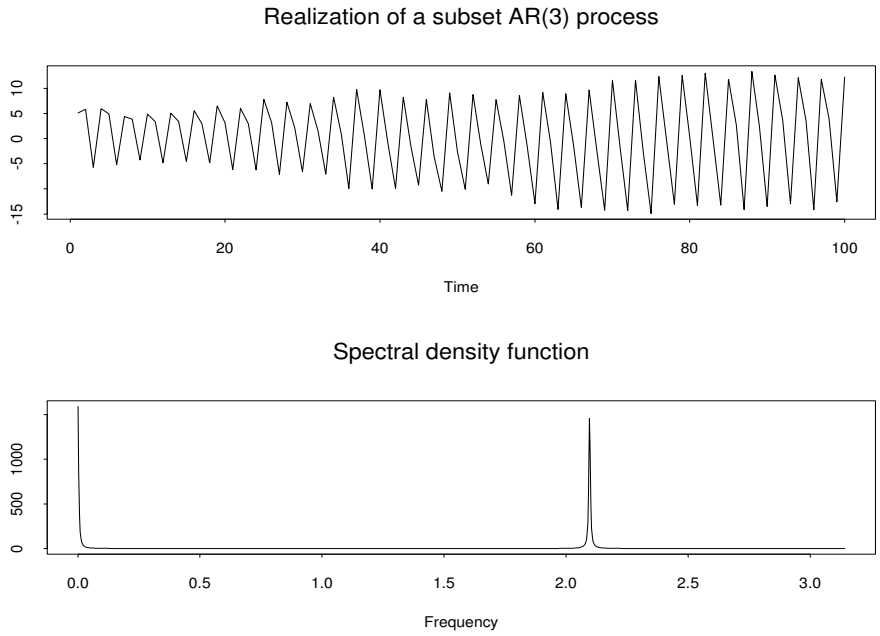
along with a plot of the spectral density function of the process on the interval $(0, \pi)$. The spectral density peaks at a frequency of $2\pi/3$ radians per unit time, which corresponds to a period of length 3.

**Fitting best subset models.** As in linear regression, one can search for the "best" subset AR/VAR model up to some maximum order, $p$. "Best" can be measured by one's favorite information criterion, such as Akaike (AIC), Bayesian (BIC), Schwarz (SIC), or even Minimum Description Length (MDL). Researchers have devised efficient algorithms to perform this search. One of the earliest attempts was made by McClave (1975), who used an algorithm adapted from linear regression. Penm and Terrell (1982), introduced an algorithm recursive in the maximum lag for best subset identification. Zhang and Terrell (1997) refine the search by inspecting certain statistics. Rather than performing an exhaustive search through all $2^p$ models, Sarkar and Sharma (1997) propose a statistical method for identifying the best subset.

Figure 2 shows the celebrated Canadian Lynx Trappings data. Ecological oscillations in predator-prey populations, mean that the logarithms of this data set are often modeled as a SAR process; a perennial favorite in the SAR literature. The lower part of the figure shows a spectral density estimator (the periodogram) for this data, which suggests the period of the oscillations to be approximately $2\pi/0.6 \approx 10.5$ years. In section 4, we will apply the algorithm of section 2 to perform an exhaustive search for the best SAR model.

For a given SVAR model order, one typically wishes to find maximum likelihood (ML) estimates of the parameters. Using standard arguments,

Figure 1: The process $X_t - 0.99X_{t-3} = Z_t$, $\{Z_t\} \sim \text{WN}(0,1)$. Top: a realization of the process with Gaussian noise. Bottom: the corresponding spectral density function.




the -2 log likelihood for the vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from the Gaussian SVAR process of dimension $d$ defined by equation (1), can be shown to be

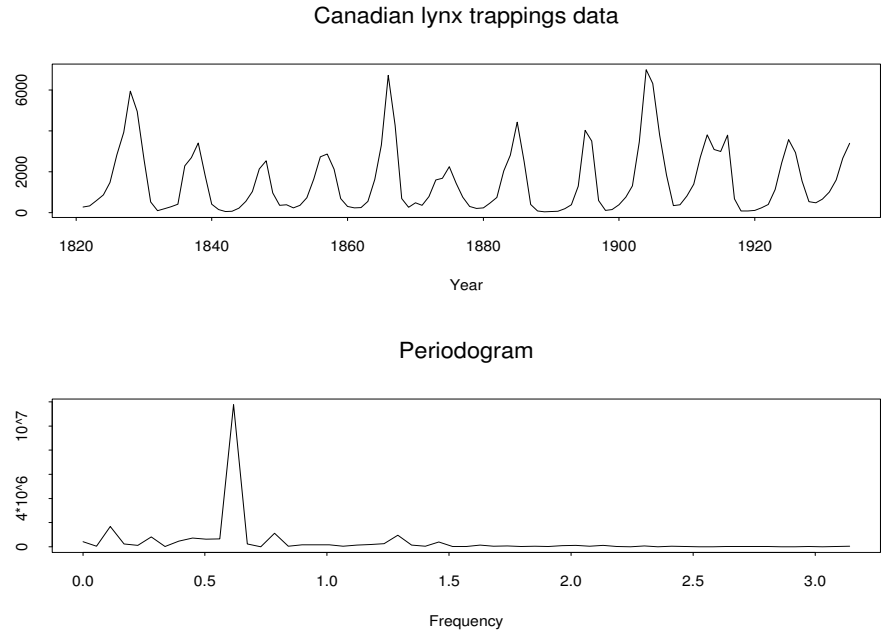
$$\begin{aligned}\mathcal{L}\left(\Phi_K(k_1), \ldots, \Phi_K(k_m), U_K\right) &= nd\log(2\pi) + \log\det(\Gamma_{k_m}) \\ &+ (n-k_m)\log\det(U_K) + \left[\mathbf{X}_1', \ldots, \mathbf{X}_{k_m}'\right]\Gamma_{k_m}^{-1}\left[\mathbf{X}_1', \ldots, \mathbf{X}_{k_m}'\right]' \\ &+ \sum_{t=k_m+1}^{n}\left[\mathbf{X}_t - \sum_{j\in K}\Phi_K(j)\mathbf{X}_{t-j}\right]' U_K^{-1}\left[\mathbf{X}_t - \sum_{j\in K}\Phi_K(j)\mathbf{X}_{t-j}\right], \quad (2)\end{aligned}$$

where $\Gamma_{k_m} = \mathbf{E}\left([\mathbf{X}_1', \ldots, \mathbf{X}_{k_m}']'[\mathbf{X}_1', \ldots, \mathbf{X}_{k_m}']\right)$.

**Remark 1**

*The potentially large number of parameters involved in ML estimation*

Figure 2: The annual Canadian lynx trappings data. Top: numbers of lynx trapped between 1821 and 1934. Bottom: the periodogram of the data.




*($d^2k_m + \frac{d^2+d}{2}$ of them), and the possible existence of many local minima which are much larger than the global minimum, makes the numerical search for the minimizers a difficult problem. The feasibility of ML estimation is therefore highly dependent upon good initial estimates.*

For this, and the reason that one may wish to avoid ML estimation altogether, it is important to consider alternative **fast** and **simple** SVAR estimation methods for obtaining models with **high likelihoods**. Recently, Brockwell, Dahlhaus, and Trindade (2002), introduced a method for doing just that. Their method, the **BDT Algorithm** which we consider in section 2, is recursive in the model order, parameter estimates of larger order models being constructed from those of smaller order models. Since the Brockwell *et al.* (2002) paper focuses mostly on theoretical aspects, the main purpose of this article is to serve as a pragmatic complement to it in the following

ways:

(i) Elucidate the recursive structure of the Algorithm.

(ii) Discuss the main issues involved in implementing the Algorithm in a high-level programming language like Fortran 90.

(iii) Provide coded versions of the Algorithm along with examples that illustrate its usage.

Section 3 accomplishes the first two goals, in the framework of a binary tree of pointer-linked nodes. The examples are presented in Section 4, which also illustrates some *meta* applications of the Algorithm. (Accompanying data sets are provided in Appendix B.) Fortran 90 programs implementing a univariate (BDT.F90) and a bivariate (BDT2.F90) version of the Algorithm are provided in Appendix C. Appendix A summarizes the function of the principal subroutines in these programs.

# 2 Estimation Methods

In this section we discuss alternative SVAR parameter estimation methods to ML. The first is a generalization of the well-known Yule-Walker method of moments estimator for full-set modeling, and has not previously appeared in the literature in this form. The second is a flexible recursive Burg-type algorithm, introduced by Brockwell *et al.* (2002), whose structure and implementation is the main focus of this paper. In order to introduce both methods, we will need to consider not only the (*forward*) SVAR model (1), but also the *backward* SVAR model

$$\mathbf{X}_t = \sum_{j \in K^*} \Psi_{K^*}(j) \mathbf{X}_{t+j} + \mathbf{Z}_t, \quad \{\mathbf{Z}_t\} \sim \mathrm{WN}(\mathbf{0}, V_{K^*}), \tag{3}$$

where $K^* = \{k_m - k_{m-1}, \ldots, k_m - k_1, k_m\}$, and suppose that the process $\{\mathbf{X}_t\}$ is *causal* (meaning that the current value of the series can be expressed as a function of current and past values of the white noise sequence as, $\mathbf{X}_t = \sum_{j=0}^{\infty} \Upsilon_j \mathbf{Z}_{t-j}$).

## 2.1 Non-Recursive Estimation: The Yule-Walker Equations

If we multiply both sides of (1) by $\mathbf{X}_{t-i}$, $i = 0, k_1, \ldots, k_m$ in turn, and (noting the causal representation) take expectations, we obtain the so-called

Yule-Walker (YW) equations:

$$\Gamma(k) = \sum_{j\in K} \Phi_K(j)\Gamma(k-j), \quad k \in K, \tag{4}$$

$$U_K = \Gamma(0) - \sum_{j\in K} \Phi_K(j)\Gamma(j)'. \tag{5}$$

For the backward SVAR model (3), the corresponding YW equations are

$$\Gamma(k)' = \sum_{j\in K^*} \Psi_{K^*}(j)\Gamma(j-k), \quad k \in K^*, \tag{6}$$

$$V_{K^*} = \Gamma(0) - \sum_{j\in K^*} \Psi_{K^*}(j)\Gamma(j). \tag{7}$$

Now define $R_K$ and $G_K$ to be matrices of autocovariances as follows: with $k_0 = 0$, define the $(i,j)$th, $i,j = 1,\ldots,m+1$, block-matrix entry of $R_K$ to be,

$$(R_K)_{(i,j)} = \begin{cases} \Gamma(k_{j-1} - k_{i-1}), & j \geq i \\ \Gamma(k_{j-1} - k_{i-1})', & j < i \end{cases},$$

and $G_K$ obtained from $R_K$ by striking out the first block row and column. The $m+1$ forward YW equations can now be succinctly written in block-matrix form as,

$$[I_d, -\Phi_K(k_1), -\Phi_K(k_2), \ldots, -\Phi_K(k_{m-1}), -\Phi_K(k_m)]\, R_K = [U_K, 0, \ldots, 0]\,,$$

and the backward YW equations as,

$$[-\Psi_{K^*}(k_m), -\Psi_{K^*}(k_m - k_1), \ldots, -\Psi_{K^*}(k_m - k_{m-1}), I_d]\, R_K = [0, \ldots, 0, V_{K^*}]\,.$$

Defining

$$\Gamma_K \equiv [\Gamma(k_1), \Gamma(k_2), \ldots, \Gamma(k_{m-1}), \Gamma(k_m)]\,,$$

and

$$\Phi_K \equiv [\Phi_K(k_1), \Phi_K(k_2), \ldots, \Phi_K(k_{m-1}), \Phi_K(k_m)]\,,$$

we can write (4)-(5) in the reduced block-matrix form

$$\Gamma_K = \Phi_K G_K, \tag{8}$$

$$U_K = \Gamma(0) - \Phi_K \Gamma_K'. \tag{9}$$

These can now be solved for $\Phi_K$ and $U_K$:

$$\Phi_K = \Gamma_K G_K^{-1}, \tag{10}$$

$$U_K = \Gamma(0) - \Gamma_K G_K^{-1} \Gamma_K', \tag{11}$$

where $G_K^{-1}$ denotes any generalized inverse of $G_K$. The solution $\Phi_K$, gives the minimum mean-squared error linear predictor of $\mathbf{X}_t$ in terms of $\mathbf{X}_{t-i},\ i \in K$. Its mean-squared error is $U_K$. Analogous results hold for the backward YW equations.

When fitting SVAR model (1) to a set of observations $\mathbf{x}_1, \ldots, \mathbf{x}_n$ from the zero-mean random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$, one of the simplest approaches is to substitute sample estimates for the autocovariances in (10) and (11). Taking the usual estimator of the autocovariance matrix at lag $h$ to be,

$$\hat{\Gamma}(h) \quad = \quad \begin{cases} \frac{1}{n}\sum_{t=1}^{n-h} \mathbf{x}_{t+h}\mathbf{x}_t' & \text{, if } h \geq 0, \\ \\ \hat{\Gamma}(-h)' & \text{, if } h < 0, \end{cases} \tag{12}$$

the resulting method of moments estimates are,

$$\hat{\Phi}_K \quad = \quad \hat{\Gamma}_K \hat{G}_K^{-1}, \tag{13}$$

$$\hat{U}_K \quad = \quad \hat{\Gamma}(0) - \hat{\Phi}_K \hat{\Gamma}_K'. \tag{14}$$

These are the so-called YW estimates in the full-set case, and we will refer to their subset generalization by the same name. The fitted YW SVAR model is therefore,

$$\mathbf{X}_t = \hat{\Phi}_K(k_1)\mathbf{X}_{t-k_1} + \cdots + \hat{\Phi}_K(k_m)\mathbf{X}_{t-k_m} + \mathbf{Z}_t, \quad \{\mathbf{Z}_t\} \sim \text{WN}(\mathbf{0}, \hat{U}_K). \tag{15}$$

## 2.2 Recursive Estimation: The BDT Algorithm

By defining the empirical forward and backward *prediction error residuals* $\hat{\boldsymbol{\varepsilon}}_K$ and $\hat{\boldsymbol{\eta}}_{K^*}$, associated with models (1) and (3) as,

$$\hat{\boldsymbol{\varepsilon}}_K(t) = \mathbf{x}_t - \sum_{i \in K} \hat{\Phi}_K(i)\mathbf{x}_{t-i}, \quad \text{and,} \quad \hat{\boldsymbol{\eta}}_{K^*}(t) = \mathbf{x}_t - \sum_{j \in K^*} \hat{\Psi}_{K^*}(j)\mathbf{x}_{t+j},$$

Brockwell *et al.* (2002), introduce a *family* of SVAR model parameter estimators, based on Burg's (1968) recursive algorithm. Their BDT Algorithm, takes the following form.

**Algorithm 1 (The BDT Algorithm)**

$$
\begin{aligned}
\hat{\Phi}_K(k_m) &= \boxed{\cdots} && (16)\\
\hat{\Phi}_K(i) &= \hat{\Phi}_J(i) - \hat{\Phi}_K(k_m)\hat{\Psi}_{J^*}(k_m - i), \quad i \in J\\
\hat{\Psi}_{K^*}(k_m) &= \hat{V}_{J^*}\hat{\Phi}_K(k_m)'\hat{U}_J^{-1} && (17)\\
\hat{\Psi}_{K^*}(j) &= \hat{\Psi}_{J^*}(j) - \hat{\Psi}_{K^*}(k_m)\hat{\Phi}_J(k_m - j), \quad j \in J^* && (18)\\
\hat{U}_K &= \hat{U}_J - \hat{\Phi}_K(k_m)\hat{V}_{J^*}\hat{\Phi}_K(k_m)'\\
\hat{V}_{K^*} &= \hat{V}_{J^*} - \hat{\Psi}_{K^*}(k_m)\hat{U}_J\hat{\Psi}_{K^*}(k_m)' && (19)\\
\hat{\boldsymbol{\varepsilon}}_K(t) &= \hat{\boldsymbol{\varepsilon}}_J(t) - \hat{\Phi}_K(k_m)\hat{\boldsymbol{\eta}}_{J^*}(t - k_m) && (20)\\
\hat{\boldsymbol{\eta}}_{K^*}(t) &= \hat{\boldsymbol{\eta}}_{J^*}(t) - \hat{\Psi}_{K^*}(k_m)\hat{\boldsymbol{\varepsilon}}_J(t + k_m) && (21)
\end{aligned}
$$

*with initial conditions,*

$$
\begin{aligned}
\hat{\boldsymbol{\varepsilon}}_\emptyset(t) &= \hat{\boldsymbol{\eta}}_\emptyset(t) = \begin{cases} \mathbf{x}_t, & t \in \{1, \ldots, n\},\\ \mathbf{0}, & \text{otherwise,}\end{cases}\\
\hat{U}_\emptyset &= \hat{\Gamma}(0) = \hat{V}_\emptyset,
\end{aligned}
$$

*and the sets $J$ and $J^*$, formed from the sets $K$ and $K^*$, respectively, by omitting $k_m$.*

A variety of different estimators can be obtained by an appropriate selection of the boxed *reflection coefficient* expression in (16). Brockwell *et al.* (2002), note that the choice

$$
\hat{\Phi}_K(k_m) = \boxed{\left(\frac{1}{n}\sum_{t=1}^{n+k_m} \hat{\boldsymbol{\varepsilon}}_J(t)\hat{\boldsymbol{\eta}}_{J^*}(t - k_m)'\right)\hat{V}_{J^*}^{-1},} \qquad (22)
$$

gives precisely the **YW estimators** (13) and (14) (reformulated via similar recursions, the resulting Algorithm is known as Levinson-Durbin), but that selecting $\hat{\Phi}_K(k_m)$ to be the minimizer of the weighted sum of forward and backward prediction errors

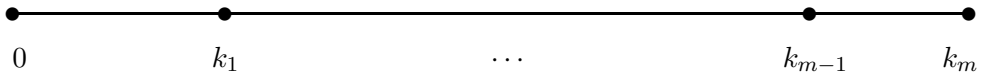
$$
\sum_{t=k_m+1}^{n} \left[\hat{\boldsymbol{\varepsilon}}_K(t)'A\hat{\boldsymbol{\varepsilon}}_K(t) + \hat{\boldsymbol{\eta}}_{K^*}(t - k_m)'B\hat{\boldsymbol{\eta}}_{K^*}(t - k_m)\right], \qquad (23)
$$

tends to produce models with consistently higher Gaussian likelihoods. By selecting different weight matrices A and B, they propose a total of three additional methods: Burg, Vieira-Morf, and Nuttall-Strand, each being a

plausible subset generalization of existing full-set analogues with the same name.

The BDT Algorithm necessarily couples together the forward and backward modeling problems. Arranging the elements of $K$ on the number line as shown in Figure 3, allows us to better visualize this coupling. The for-

Figure 3: The set of lags $K = \{k_1, \ldots, k_m\}$ arranged on the number line.




ward set of lags, $K$, are simply the distances of the elements of $K$ from the origin; while the backward set of lags, $K^*$, are the corresponding distances from $k_m$.

Note that the YW estimator, $\hat{\Phi}_K(k_m)$, obtained from (13), requires the inversion of $\hat{G}_K$, which is of dimension $md$. Recursive algorithms are better suited to searching for a best subset model with a specified maximum number of lags, and involve inversion of matrices whose dimension is at most $d$ ($d^2$ in some instances).

# 3 Implementing the BDT Algorithm

Apart from the special case of YW, estimators arising from the BDT Algorithm cannot in general be reformulated in a non-recursive manner. The intricate structure of the recursions, a by-product of the forward and backward model coupling, can seem rather daunting from a programming perspective. In this section therefore, we discuss the main issues involved in implementing this Algorithm in a high-level programming language like Fortran 90. An important goal is to minimize computing time, and our approach will be to create a linked list of nodes in the form of a binary tree. In the process, we will make use of recursive pointers, recursive subroutines, and data types that incorporate recursive definitions.

## 3.1 Building a binary tree of linked nodes

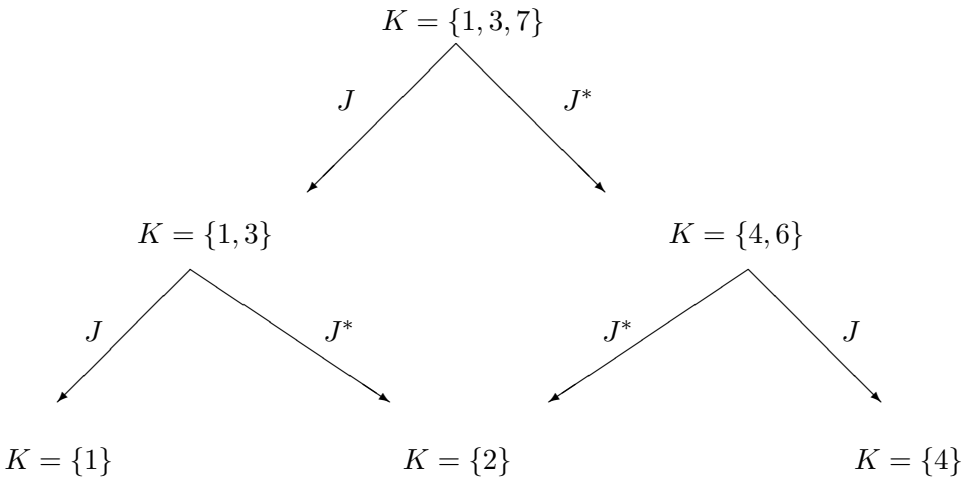
The recursive solution of the equations defining the BDT Algorithm, generates a collection of estimators of SVAR models of increasing orders, until the required order is reached. Suppose modeling on the set of lags $\{1, 3, 7\}$

is desired. To determine where application of the algorithm should begin, we first need to work down to derived subsets of lags comprised of just one lag. This is done by successively forming the $J$ and $J^*$ subsets of lags for each *parent* set of lags $K$, as shown in Figure 4. Each of the subsets $J$ and $J^*$ then assumes the role of a parent lag, $K$, and the procedure is repeated. In the resulting binary tree structure, we will refer to all the modeling information pertaining to a set of lags as a *node*. The number of lags in a node will define its *level* in the tree. The strategy for this recursive tree-building will then be as follows:

**Pseudocode 1 (Build Tree)**

*level* := $m$
while *level* $> 1$ do
    for *each node in current level* do
        *compute lags in* $J$ *and* $J^*$ *subnodes*
        *direct pointers to* $J$ *and* $J^*$ *subnodes* od
    *level* := *level* $- 1$ od

Figure 4: Binary tree of linked nodes for modeling on the set of lags $\{1, 3, 7\}$.



For programming, it will be necessary to rewrite the backward model equations (17), (18), (19), and (21), in the unstarred lags format:

$$\hat{\Psi}_K(k_m) = \hat{V}_J \hat{\Phi}_{K^*}(k_m)' \hat{U}_{J^*}^{-1}, \tag{24}$$

$$\hat{\Psi}_K(j) = \hat{\Psi}_J(j) - \hat{\Psi}_K(k_m)\hat{\Phi}_{J^*}(k_m - j), \quad j \in J, \tag{25}$$

$$\hat{V}_K = \hat{V}_J - \hat{\Psi}_K(k_m)\hat{U}_{J^*}\hat{\Psi}_K(k_m)', \tag{26}$$

$$\hat{\boldsymbol{\eta}}_K(t) = \hat{\boldsymbol{\eta}}_J(t) - \hat{\Psi}_K(k_m)\hat{\boldsymbol{\varepsilon}}_{J^*}(t + k_m), \tag{27}$$

obtained by noting that $(K^*)^* = K$, and $(J^*)^* = J$. The unstarred lags representation carifies how the backward model estimates should be computed for any given node. Letting `node` be a user-defined derived data type, and $K = \{k_1, \ldots, k_m\}$ denote a generic set of lags for a given node, this data type should then consist of the following components:

**(i)** `lags -` vector containing the current set of lags, $K$, on which modeling is desired (type `integer`).

**(ii)** `level -` scalar specifying the level of the node in the tree (type `integer`).

**(iii)** $\Phi_K$, $\Psi_K$ `-` vectors of forward and backward model coefficient matrices; that is $\Phi_K = \{\Phi_K(k_1), \ldots, \Phi_K(k_m)\}$, and $\Psi_K = \{\Psi_K(k_1), \ldots, \Psi_K(k_m)\}$ (type `real`).

**(iv)** $U_K$, $V_K$ `-` estimates of the white noise covariance matrices for the forward and backward modeling problems (type `real`).

**(v)** $\boldsymbol{\varepsilon}_K(t)$, $\boldsymbol{\eta}_K(t)$ `-` vectors of prediction error residuals for the forward and backward modeling problems (type `real`).

**(vi)** `reg, str -` pointers to the $J$ and $J^*$ subnodes, respectively, one level below the current level (type `node`, defined recursively).

The tree can be linked by recursive calls to a `RECURSIVE SUBROUTINE`, with `level, lags`, and a `pointer` of type `node` as arguments. This routine should also set a flag to signal when a particular node has been initialized (linked in the list), but the remaining components, (iii)-(vi), not yet evaluated (node *unfilled*). The flag, setting the first row and column entry of $U_K$ to zero for example, will be used by a subsequent node-filling routine. Two pointers should emanate from each node, `reg` pointing to $J$, and `str` to $J^*$. At level 1, these pointers should point nowhere (`NULLIFY`). From Figure 4, we note that both nodes at level 2 have the set $\{2\}$ as their $J^*$ subnode. Two copies of this subnode can be made, each linked to its appropriate parent node. This duplication of nodes can be avoided by a more complex program, since otherwise exactly $2^m$ nodes are created for modeling on $m$ initial lags.

### 3.2 Filling the nodes

Once the tree with all appropriate linking pointers is in place, we will need to evaluate the remaining components, (iii)-(vi), of each node. This can be done by "walking" through the tree, following the linked list of nodes. Once again, a `RECURSIVE SUBROUTINE` taking a pointer as argument can be employed to achieve this, certifying first that each node has not yet been filled by checking the flag alluded to earlier.

The recursion should be implemented in such a way that the tree is walked to level 1 where the node-filling can begin. From Figure 4, we note that this involves filling nodes $\{1\}$, $\{2\}$, and $\{4\}$ first. With these, we can now fill nodes $\{1,3\}$ and $\{4,6\}$, at level 2. The operation terminates at the top node, $\{1,3,7\}$, if the pointer to this node is passed as the original argument to the recursive node-filling subroutine. The pseudo-code for this phase of the implementation could therefore be:

**Pseudocode 2 (Fill Tree)**

```
call node filling routine with pointer to top node as argument
while level of current node > 1  do
      call node filling routine with reg pointer as argument
      if current node unfilled  then fill it  fi
      call node filling routine with str pointer as argument
      if current node unfilled  then fill it  fi od
if level of current node = m then fill top node  od
```

This coding will give filling-precedence to nodes at low levels that emanate from parents with respect to which they are $J$ subnodes. In example 4, this would result in the following filling order: $\{1\}$, $\{2\}$ ($J^*$ subnode of $\{1,3\}$), $\{1,3\}$, $\{4\}$, $\{2\}$ ($J^*$ subnode of $\{4,6\}$), $\{4,6\}$, $\{1,3,7\}$.

Note that in the univariate case there is no distinction between forward and backward model parameters for the same set of lags; that is, the YW equations give $\hat{U}_K \equiv \hat{V}_K$, and $\hat{\Phi}_K \equiv \hat{\Psi}_K$, for any set $K$. Both this and the fact that all parameters are scalars, greatly simplifies the programming task when $d = 1$.

## 4 Examples

Included in Appendix C are BDT.F90 and BDT2.F90. These are, respectively, univariate and bivariate Fortran 90 programs implementing the BDT

Algorithm. The programs utilize a few linear algebra subroutines in the International Mathematical and Statistical Library (IMSL). In this section we document how to run the programs in order to fit a particular SAR/SVAR model to a given data set, and illustrate some potential *meta* applications that involve repeated modeling with each of the programs in the inner loop. (We have not provided the programs for Examples 3-5, but they are available from the author upon request.)

### 4.1 Example 1: Running BDT.F90

In order to fit the model

$$X_t = \phi_K(1)X_{t-1} + \phi_K(2)X_{t-2} + \phi_K(3)X_{t-3} + \phi_K(4)X_{t-4} + \phi_K(10)X_{t-10} + \phi_K(11)X_{t-11} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

of Table 2 to the mean-corrected base 10 logarithms lynx data (lynx10.tsm in Appendix B) of Figure 2, we ran the compiled version of BDT.F90 with the following inputs at the prompt (>):

```
 %%%%%%%%%%%%%% Univariate SAR Modeling Program %%%%%%%%%%%%%%

     File name of time series for modelling:
> lynx10.tsm
 Do you wish to mean-correct the observations (1=yes, 0=no)?
> 1
 There are 114 observations.
 First obs is -0.47391147326671135 last is 0.6273039283027888
 Enter the number of lags to be modeled (<27):
> 6
 Enter the lags:
> 1 2 3 4 10 11
 Enter the method for obtaining the reflection coefficients:
 Yule-Walker (1), Burg (2), Vieira-Morf (3), Nuttall-Strand (4):
> 2
 ************************************************
 The estimated subset Burg AR coefficients are:
 Phi( 1):   1.15639
 Phi( 2):  -0.50191
 Phi( 3):   0.19869
 Phi( 4):  -0.21127
 Phi(10):   0.37899
```

```
 Phi(11):  -0.42454
 ************************************************
 Burg WN variance estimate : 3.61762021546651741E-2
 RSS/n WN variance estimate: 3.69130827522938937E-2
 -2 Log Likelihood (Burg) : -46.962408999128101
 -2 Log Likelihood (RSS/n): -46.985742242956121
 AICC (RSS/n)  : -31.929138811254461
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

This gives the Burg estimators. The Yule-Walker estimates can be obtained by selecting "1" in the last step. The remaining estimators proposed by Brockwell *et al.* (2002), can be obtained by selecting "3" (Vieira-Morf), and "4" (Nuttall-Strand). The RSS/n is the MLE of $\sigma^2$ for the given set of SAR coefficient estimates, $\{\hat{\phi}_K(k_1), \ldots, \hat{\phi}_K(k_m)\}$, thus it comes as no surprise that the -2 Log Likelihood based on it is never larger than the -2 Log Likelihood based on the Burg $\hat{\sigma}^2$. The AICC is a bias-corrected version of AIC; see Brockwell and Davis (1991).

### 4.2 Example 2: Running BDT2.F90

To fit the bivariate SVAR model

$$\mathbf{X}_t = \Phi_K(1)\mathbf{X}_{t-1} + \Phi_K(3)\mathbf{X}_{t-3} + \mathbf{Z}_t, \quad \{\mathbf{Z}_t\} \sim \mathrm{WN}(\mathbf{0}, U_K),$$

to the sun2.tsm data (Appendix B), we ran the compiled version of BDT2.F90 with the following inputs at the prompt (>):

```
 %%%%%%%%%%%%%% Bivariate SVAR Modeling Program %%%%%%%%%%%%%%

     Enter file name of time series for modelling:
>sun2.tsm
 Do you wish to mean-correct the observations (1=yes, 0=no)?
>1
 There are           50  observations.
 First obs is              53.5200 last is             35.6200
 First obs is             -10.4800 last is             27.6200
 Enter the number of lags to be modeled (<27):
>2
 Enter the lags:
>1 3
 Is the true white noise covariance matrix known (1=yes, 0=no)?
```

```
>0
 Enter the method for obtaining the reflection coefficients.
 Yule-Walker (1), Burg (2), Vieira-Morf (3), Nuttal-Strand (4):
>3
 ************************************************
 Estimated subset Morf coefficient matrices:
 Phi(           1 ):
            -0.853995              1.571658
            -0.913452              1.279817
 Phi(           3 ):
             0.029511              0.092263
             0.291517             -0.150232

 ************************************************
 Estimated Morf (forward) WN covariance matrix:
           145.678543            220.305063
           220.305063            580.954041

 -2 Log Like (Morf) :    812.877439308433
 -2 Log Like (RSS/n):    812.820447412800

 Estimated RSS/n (forward) WN covariance matrix:
           143.844793            221.765063
           221.765063            598.346541
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Like the univariate program, two estimates of $U_K$ are given: the first is method-specific (method 3, Vieira-Morf, in this case), the second, RSS/n, is the MLE of $U_K$ holding $\Phi_K(1)$ and $\Phi_K(3)$ fixed at their estimated values. In the multivariate case, there is no known closed form for the RSS/n estimate like there was in the univariate case (see Appendix A.6). If the data was obtained via simulation and a "1" was entered at the 5th prompt, the program would use the furnished value of $U_K$ as the initial guess for RSS/n in the optimizing routine. Since we entered "0", the program will use the Morf WN estimate as the initial guess.

### 4.3 Example 3: Best Subset Searching

The lynx data of Figure 2 is often cited in the literature in connection with SAR modeling. Using YW estimation and their own respective non-exhaustive search algorithms, Tong (1977), Penm and Terrell (1982), Zhang

and Terrell (1997), and others, identify a SAR(1,2,4,10,11), i.e. $K = \{1, 2, 4, 10, 11\}$, as the best SAR model according to a variety of information criteria. It is important to realize that some of these search methods are non-exhaustive and statistical in nature, and will therefore not guarantee a correct identification with certainty.

Using the BDT Algorithm, we performed an exhaustive search for the minimum AICC SAR model, for the mean-corrected base 10 logarithms of the lynx data. Letting $p$ denote the maximum lag considered in the search (meaning that $2^p$ models had to be checked), we considered $p = 4$, 8, and 12, in turn. This set of searches was performed for each of the YW and Burg estimation methods. The number of subsets out of the $2^p$ that resulted in non-causal fitted models, as well as the corresponding computational (CPU) times taken by each search, were recorded. The AICC (RSS/n) of the best SAR model was also computed.

Table 1: Results of best (minimum AICC) SAR model search for the mean-corrected base 10 logarithms of the Canadian lynx data of Figure 2.

| Estimation method | $p$ | Lags in best subset | AICC of best model | Prop. of non causal models | CPU time (secs) |
|---|---|---|---|---|---|
| YW | 4 | 1,2,4 | -9.89 | 3/16 | 1.2 |
| | 8 | 1,2,4,8 | -16.17 | 78/256 | 21.3 |
| | 12 | 1,2,4,10,11 | -31.80 | 1392/4096 | 637.2 |
| Burg | 4 | 1,2,4 | -10.08 | 3/16 | 1.8 |
| | 8 | 1,2,4,8 | -16.27 | 81/256 | 29.6 |
| | 12 | 1,2,3,4,10,11 | -31.93 | 1489/4096 | 678.7 |

The results are summarized in Table 1. The best SAR model with maximum lag 12 found by the YW method, coincides with that identified by other researchers as discussed above; but that arrived at by the new subset Burg method, adds lag 3, and has a slightly lower value of AICC. Note also that the proportion of subsets resulting in non-causal fitted models (meaning that a SAR model with these lags is inappropriate for the data), remained steady at approximately 1/3 across all searches. At the same value of $p$, CPU times for Burg are slightly higher than those for YW, both growing exponentially with $p$. The computations were carried out on a Sun Enter-

prise 450 unix server, equipped with about 4G of memory. The most severe limiting factor in this type of computation is available memory, since at least $2^p$ pointers have to be `allocated`.

In Table 2, we present the parameter estimates of the best SAR model identified by each respective method when $p = 12$. The constrained ML estimates were obtained via the ITSM2000 package (Brockwell and Davis, 2002), starting with the Burg estimates.

Table 2: Best SAR models fitted to the mean-corrected base 10 logarithms of the Canadian lynx data, as identified by each respective method. The Maximum Likelihood estimates were obtained by starting with the Burg estimates, and constraining the ML search to the same SAR lags.

| Parameter | Estimates by Method | | |
|---|---|---|---|
| | Maximum Likelihood | Burg | Yule-Walker |
| $\phi_K(1)$ | 1.148 | 1.156 | 1.094 |
| $\phi_K(2)$ | −0.502 | −0.502 | −0.357 |
| $\phi_K(3)$ | 0.199 | 0.199 | |
| $\phi_K(4)$ | −0.217 | −0.211 | −0.127 |
| $\phi_K(10)$ | 0.351 | 0.379 | 0.324 |
| $\phi_K(11)$ | −0.401 | −0.425 | −0.362 |
| $\sigma^2$ | 0.037 | 0.037 | 0.038 |
| AICC | −32.22 | −31.93 | −31.80 |

### 4.4 Examples 4 and 5: Adaptive Behavior

In these simulated bivariate examples, we illustrate the component-wise convergence of the Burg and YW estimates to their true values, as a function of observation number or time. This adaptive behavior is important in on-line applications where it is desirable to monitor the convergence of the estimates, particularly when observation number is still low. Note however that the BDT Algorithm is recursive in the model order, not observation number, and is therefore not truly adaptive in that sense. The entire Algorithm needs to be re-run from scratch whenever a new observation becomes available.

The *characteristic polynomial* of SVAR model (1) is

$$P(z) = \det\left[I_d - \Phi_K(k_1)z^{k_1} - \cdots - \Phi_K(k_m)z^{k_m}\right].$$

The model is causal if the zeroes of its characteristic polynomial are all greater than one in magnitude. It is well-known that in the univariate full-set case, the YW estimators can be severely biased if the roots of the AR characteristic polynomial are close to the unit circle (quasi-non-stationarity). To allow for the expected dependence of performance on the location of the zeroes of $P(z)$, we considered causal models with different configurations of these zeroes. A total of 250 observations were sequentially simulated from the basic SVAR(2) model,

$$\mathbf{X}_t = \Phi\mathbf{X}_{t-2} + \mathbf{Z}_t \equiv \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \mathbf{X}_{t-2} + \mathbf{Z}_t, , \quad \mathbf{Z}_t \sim \mathrm{N}_2(\mathbf{0}, I_2),$$

and we started estimation at observation number 10.

**Example 4**

$$\Phi = \begin{bmatrix} 0.547 & -0.300 \\ 0.700 & -0.457 \end{bmatrix}, \qquad P(z) = (1 - 0.25z^2)(1 + 0.16z^2),$$

with roots of characteristic polynomial: $\pm 2$, $\pm 2.5i$.

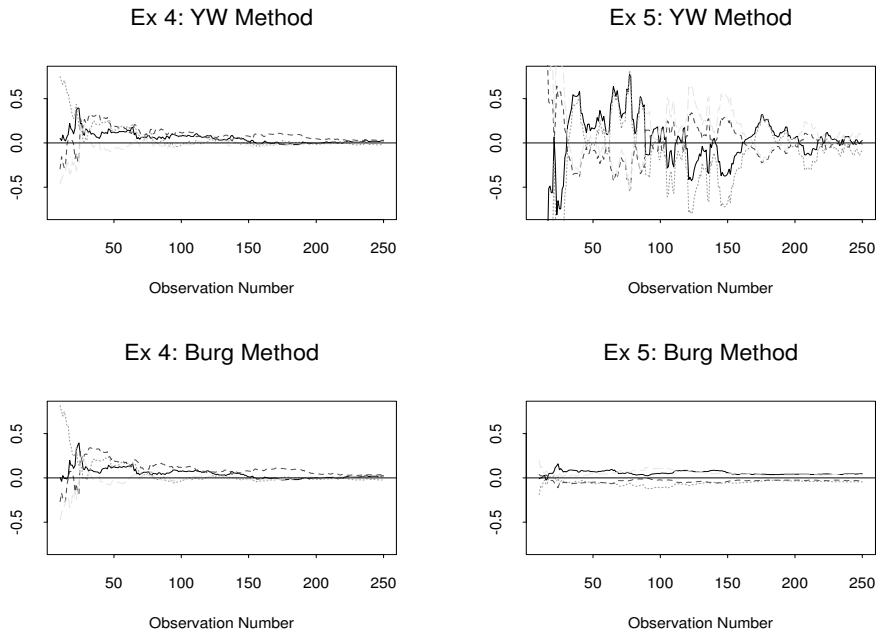
**Example 5**

$$\Phi = \begin{bmatrix} 1.414 & -0.300 \\ 0.700 & 0.497 \end{bmatrix}, \qquad P(z) = (1 - 0.98^2z^2)(1 - 0.95^2z^2),$$

with roots of characteristic polynomial: $\pm 1.02$, $\pm 1.03$.

The results are displayed in Figure 5, where we plot the component-wise departures of the estimated SVAR coefficient matrices from their true values, $\Phi_{ij} - \hat{\Phi}_{ij}$, $i, j = 1, 2$, as a function of observation number. The pattern of convergence between the two methods is similar in Example 4, but dramatically different in Example 5. Although based on a single simulated realization presented only to illustrate a meta application of the bivariate BDT Algorithm, this phenomenon is nevertheless consistent with what has been noted about the behavior of YW versus Burg in quasi-non-stationary modeling. Since both estimators and the MLE all have the same asymptotic distribution, there is little cause for concern with large samples; it is with small samples that one should exercise caution when selecting an estimation method. The more extensive analysis by Brockwell *et al.* (2002), suggests that Burg is in general a better estimator than YW.

Figure 5: Departures of the 4 estimated components of the SVAR coefficient matrix from their true values, by method, for the simulated realizations of Examples 4 and 5.




# 5 Conclusion

We have discussed the popularity of multivariate subset autoregressive models, and highlighted the importance of fast, simple, and efficient methods for the estimation of their parameters. One such set of estimators is obtained via the classical Yule-Walker method-of-moments, which we have presented as the solution to a system of simultaneous linear matrix equations. A recently introduced more general estimation method, the BDT Algorithm, is recursive in the order of the fitted model, thus avoiding the (potentially large) matrix inversions required in solving the Yule-Walker equations. By suitably modifying the reflection coefficient calculation, this Algorithm can produce a variety of estimators with different finite sample properties, among them Yule-Walker. We have illustrated the recursive structure of this Algorithm,

and discussed its implementation in a high-level programming language like Fortran 90. The speed of the Algorithm was assessed in finding a best subset model for the Canadian lynx data, and shown, in problems of moderate size, to be a feasible alternative to non-exhaustive search techniques which do not guarantee correct subset identification. We concluded with two simulated bivariate examples that illustrate the adaptive performance of the Yule-Walker and Burg estimators, implemented via the BDT Algorithm. We find that the Burg estimates tend to stabilize more quickly than Yule-Walker, and are far less affected by proximity of the model to non-stationarity.

## 6 Acknowledgements

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# A Description of Principal Program Subroutines

As already stated, the core of the subset modeling programs `Burg` and `Burg2` is the globally visible `MODULE Tree`, with `SUBROUTINE Make_Tree` its driving subroutine. In this section, we will provide a brief description of the essential functions of its main constituent subroutines.

## A.1 `Build_Node_Tree`

This is a `RECURSIVE SUBROUTINE` that initializes the tree of nodes by allocating pointers to and from nodes. It takes on the `level`, `lags`, and a `pointer` of type `node` as arguments. It begins execution at the unique node of level $m$ (`top_node`), creating pointers to the $J$ and $J^*$ subnodes (`this_node%reg` and `this_node%str`, respectively). Following these pointers to level $m-1$, `Build_Node_Tree` subsequently allocates pointers to the subnodes in level $m-2$. It achieves this by calling itself with the appropriate arguments:`level` should be the current level minus one, and pointers `this_node%reg` and `this_node%str`. The procedure is repeated, always following pointer `this_node%reg` before `this_node%str`, until level 1 is reached. At this point, the two pointers are initialized and made to point nowhere (`NULLIFIED`). By the order of precedence inherent in it, the routine then backs up one level and proceeds to follow pointer `this_node%str` to the "dead end" at level 1.

In this fashion, the tree is initialized from left ($J$) to right ($J^*$), with the pointer to the subnode $J^*$ of the rightmost node being allocated last. If we refer back to figure 4, the nodes for the tree of this example will be initialized in the following order:

$$\{1,3,7\} \rightarrow \{1,3\} \rightarrow \{1\} \rightarrow \{2\} \rightarrow \{4,6\} \rightarrow \{4\} \rightarrow \{2\}.$$

In order for subsequent routines to identify an initialized but unfilled (constituents of `node` empty) node, `Build_Node_Tree` will set `this_node%v` (`this_node%vf%mat(1,1)` in BDT2.F90) to zero, upon allocation of pointers.

### A.2 `Fill_Tree`

A `RECURSIVE SUBROUTINE`, taking on a `pointer` of type `node` as argument. Its function is to traverse the now initialized tree, and using the flag for an unfilled node, fill it by calling `Fill_Node`.

### A.3 `Fill_Node`

A `RECURSIVE SUBROUTINE`, called by `Fill_Tree`, whose function is to fill the particular node that its `pointer` argument points to. It is in this routine that the BDT Algorithm proper is applied, modifying the reflection coefficient calculation according to the selected method. Care must be taken when calculating the forward and backward prediction errors, $\boldsymbol{\varepsilon}_K(t)$ and $\boldsymbol{\eta}_K(t)$, before termination of the routine. Each should be calculated over a sufficiently large range of $t$ values ($1 \leq t \leq n + k_m$ for Yule-Walker, and $1 + k_m \leq t \leq n$ for the remaining methods), since subsequent nodes may use them.

### A.4 `Print_Node_Tree`

With its `pointer` argument, the `RECURSIVE SUBROUTINE Print_Node_Tree` will traverse the now completed tree of nodes, and proceed to print the estimated coefficients and white noise variance stored in each node.

### A.5 `Causal_Check`

This routine is needed in the bivariate program only, in order to ensure the obtained VAR model is causal before proceeding with the likelihood calculations. In the univariate program, this function is performed within

the likelihood calculation routine itself. The strategy is to use the state space representation to write a VAR(p) as a VAR(1), as follows:

Random vectors $\{\mathbf{X}_t, \ldots, \mathbf{X}_{t-k_m}\}$ from model (1), will satisfy the relationships

$$\begin{bmatrix} \mathbf{X}_t \\ \mathbf{X}_{t-1} \\ \mathbf{X}_{t-2} \\ \\ \vdots \\ \\ \\ \mathbf{X}_{t-k_m+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & k_1 & \cdots & k_m \\ I_d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & I_d & & & & & 0 \\ & & \ddots & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & \ddots & & \\ 0 & & & \cdots & & I_d & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t-1} \\ \mathbf{X}_{t-2} \\ \mathbf{X}_{t-3} \\ \\ \vdots \\ \\ \\ \mathbf{X}_{t-k_m} \end{bmatrix} + \begin{bmatrix} \mathbf{Z}_t \\ \mathbf{0} \\ \mathbf{0} \\ \\ \vdots \\ \\ \\ \mathbf{0} \end{bmatrix},$$

which can be written in the compact form

$$\underbrace{\mathbf{Y}_t}_{(dk_m \times 1)} = \underbrace{A}_{(dk_m \times dk_m)} \underbrace{\mathbf{Y}_{t-1}}_{(dk_m \times 1)} + \underbrace{\mathbf{W}_t}_{(dk_m \times 1)} . \tag{28}$$

In block matrix form, vectors $\mathbf{Y}_t$ and $\mathbf{W}_t$ have length $k_m$, while the square matrix $A$ has dimension $k_m$. Note that the only nonzero entries of the first block matrix row of $A$ are $\{\Phi_K(k_1), \Phi_K(k_2), \ldots, \Phi_K(k_{m-1}), \Phi_K(k_m)\}$, occurring at block matrix column numbers $\{k_1, k_2, \ldots, k_{m-1}, k_m\}$, respectively. The covariance matrix of $\mathbf{W}_t$ is

$$\Sigma_W = \mathbf{E} \begin{bmatrix} \mathbf{Z}_t \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \left[\mathbf{Z}_t', \mathbf{0}', \ldots, \mathbf{0}'\right] = \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix},$$

where we use $\Sigma$ in place of $U_K$. (28) is now a VAR(1) of dimension $dk_m$, and its causality (and thus that of the original process) can be assessed by determining if all eigenvalues of $A$ are less than 1 in absolute value.

### A.6 `Likelihood/Approx_Likelihood`

In the univariate program, we compute the exact likelihood in `SUBROUTINE Likelihood`. The only sizeable difficulty is in evaluating the model autocovariances $\gamma(0), \ldots, \gamma(k_m)$, accomplished by inverting the Yule-Walker equations. The $-2$ log likelihood, $\mathcal{L}(\boldsymbol{\phi}_K, \sigma^2)$, for the data $\mathbf{x}_1, \ldots, \mathbf{x}_n$, is

then evaluated via the Innovations Algorithm (Brockwell and Davis, 1991, prop. 5.2.2, and equation 8.7.4):

$$\mathcal{L}(\boldsymbol{\phi}_K, \sigma^2) = n\log(2\pi\sigma^2) + \sum_{t=1}^{n}\log(r_{t-1}) + \frac{1}{\sigma^2}\sum_{t=1}^{n}(\mathbf{x}_t - \hat{\mathbf{x}}_t)^2/r_{t-1}.$$

In the bivariate program, `SUBROUTINE Likelihood` uses the same approach to compute the likelihood, i.e. the Multivariate Innovations Algorithm (Brockwell and Davis, 1991, prop. 11.4.2, and equation 11.5.5):

$$\mathcal{L}(\Phi_K, \Sigma) = nd\log(2\pi) + \sum_{t=1}^{n}\log \mid V_{t-1} \mid + \sum_{t=1}^{n}(\mathbf{X}_t - \hat{\mathbf{X}}_t)' V_{t-1}^{-1}(\mathbf{X}_t - \hat{\mathbf{X}}_t).$$

Computing the model autocovariance matrices, $\Gamma(1-k_m), \ldots, \Gamma(0), \ldots, \Gamma(k_m - 1)$, is a much more formidable task here, but this can be accomplished via the state space formulation of the previous subsection. Transforming the SVAR($K$) to the VAR(1) of equation (28), gives the following solution for the autocovariances $\Gamma_Y(\cdot)$ of the process $\{\mathbf{Y}_t\}$:

$$\Gamma_Y(h) = \begin{cases} A\Gamma_Y(h)A' + \Sigma_W, & h = 0 \\ A\Gamma_Y(h-1), & h > 0 \end{cases},$$

whence we obtain

$$vec\left(\Gamma_Y(0)\right) = \left[I_{d^2k_m^2} - A \otimes A\right]^{-1} vec\left(\Sigma_W\right).$$

The required autocovariance matrices can be found in the first block row and column of the $(k_m \times k_m)$ block matrix $\Gamma_Y(0)$, since

$$\underbrace{\Gamma_Y(0)}_{(dk_m \times dk_m)} = \begin{bmatrix} \Gamma(0) & \Gamma(1) & \cdots & \Gamma(k_m - 1) \\ \Gamma(-1) & \Gamma(0) & \cdots & \Gamma(k_m - 2) \\ \vdots & & \ddots & \vdots \\ \Gamma(1 - k_m) & \cdots & \Gamma(-1) & \Gamma(0) \end{bmatrix}.$$

Due to the computational intensity involved in finding $\Gamma_Y(\cdot)$ however, the bivariate routine `Likelihood` is extremely slow. We opt instead to approximate the autocovariances via the causal representation

$$\Gamma(h) = \sum_{j=0}^{\infty}\Psi_{h+j}\Sigma\Psi_j',$$

truncating the summation at 100 terms, and computing the likelihood via (2). This "approximate likelihood", is computed in `SUBROUTINE Obj_Fun`. `SUBROUTINE Approx_Likelihood` not only calls `Obj_Fun` in order to compute this approximate likelihood for $\Sigma_{AL}$, but also searches for the white noise covariance matrix that maximizes the likelihood for the given VAR coefficient matrices ($\Sigma_{ML}$). It does so by using $\Sigma_{AL}$ as an initial guess, and by repeated calls to `SUBROUTINE Hooke`, which employs a direct search algorithm to locate the global minimum of an objective function of several variables (Hooke and Jeeves, 1961).

# B Data Sets

## B.1 lynx10.tsm

```
2.42975228000241
2.50650503240487
2.76715586608218
2.94001815500766
3.16879202031418
3.45040308615537
3.59417147911491
3.77400573025821
3.69460519893357
3.4111144185509
2.71850168886727
1.99122607569249
2.26481782300954
2.4456042032736
2.61172330800734
3.35888620440587
3.42894429003557
3.53262700122889
3.2610248339924
2.61172330800734
2.17897694729317
1.65321251377534
1.83250891270624
2.32837960343874
2.73719264270474
3.01410032151962
```

3.32817566143832
3.40414924920969
2.98091193777684
2.55750720190566
2.57634135020579
2.35218251811136
2.55630250076729
2.86391737695786
3.2143138974244
3.43536650661266
3.45803319249651
3.32613095671079
2.83505610172012
2.47567118832443
2.37291200297011
2.38916608436453
2.7419390777292
3.21031851982623
3.51995918075207
3.82743389540078
3.62879748556671
2.83695673705955
2.40654018043395
2.67486114073781
2.55388302664387
2.89431606268444
3.20248831706009
3.22427401429426
3.35237549500052
3.15411952551585
2.87852179550121
2.47567118832443
2.30319605742049
2.35983548233989
2.67117284271508
2.8668778143375
3.31005573775089
3.44886084560744
3.64650175003161
3.39984671271292

2.58994960132571
1.86332286012046
1.5910646070265
1.69019608002851
1.77085201164214
2.27415784926368
2.57634135020579
3.11126251365907
3.60541279815305
3.5434471800817
2.76863810124761
2.02118929906994
2.1846914308176
2.58771096501891
2.87966920563205
3.11627558758054
3.53970323894783
3.84453930212901
3.80023578932735
3.57909732655264
3.26387267686522
2.53781909507327
2.58206336291171
2.90741136077459
3.14238946611884
3.4334497937616
3.57978359661681
3.4900990050633
3.47494433546539
3.57863920996807
2.82865989653532
1.90848501887865
1.90308998699194
2.03342375548695
2.35983548233989
2.60097289568675
3.05384642685225
3.3859635706007
3.55315454816963
3.46760810558363

```
3.18667386749974
2.72345567203519
2.68574173860226
2.8208579894397
3
3.20139712432045
3.42439155441028
3.53096768157191
```

### B.2 sun2.tsm

```
      101.000000000000000       82.000000000000000
       66.000000000000000       35.000000000000000
       31.000000000000000        7.000000000000000
       20.000000000000000       92.000000000000000
      154.000000000000000      125.000000000000000
       85.000000000000000       68.000000000000000
       38.000000000000000       23.000000000000000
       10.000000000000000       24.000000000000000
       83.000000000000000      132.000000000000000
      131.000000000000000      118.000000000000000
       90.000000000000000       67.000000000000000
       60.000000000000000       47.000000000000000
       41.000000000000000       21.000000000000000
       16.000000000000000        6.000000000000000
        4.000000000000000        7.000000000000000
       14.000000000000000       34.000000000000000
       45.000000000000000       43.000000000000000
       48.000000000000000       42.000000000000000
       28.000000000000000       10.000000000000000
        8.000000000000000        2.000000000000000
  0.000000000000000E+000        1.000000000000000
        5.000000000000000       12.000000000000000
       14.000000000000000       35.000000000000000
       46.000000000000000       41.000000000000000
       30.000000000000000       24.000000000000000
       16.000000000000000        7.000000000000000
        4.000000000000000        2.000000000000000
        8.000000000000000       17.000000000000000
       36.000000000000000       50.000000000000000
```

```
 62.000000000000000       67.000000000000000
 71.000000000000000       48.000000000000000
 28.000000000000000        8.000000000000000
 13.000000000000000       57.000000000000000
122.000000000000000      138.000000000000000
103.000000000000000       86.000000000000000
 63.000000000000000       37.000000000000000
 24.000000000000000       11.000000000000000
 15.000000000000000       40.000000000000000
 62.000000000000000       98.000000000000000
124.000000000000000       96.000000000000000
 66.000000000000000       64.000000000000000
 54.000000000000000       39.000000000000000
 21.000000000000000        7.000000000000000
  4.000000000000000       23.000000000000000
 55.000000000000000       94.000000000000000
 96.000000000000000       77.000000000000000
 59.000000000000000       44.000000000000000
 47.000000000000000       30.000000000000000
 16.000000000000000        7.000000000000000
 37.000000000000000       74.000000000000000
```

# C Coded Versions of the BDT Algorithm

## C.1 Code for BDT.F90

```
MODULE tree

! Here we define the data type NODE which will contain
  TYPE node
    INTEGER               :: level         ! level in tree: top=m, bottom=1
    INTEGER               :: lags(26)      ! lags for node are stored here
  DOUBLE PRECISION        :: phi(26)       ! coefficients for node
  DOUBLE PRECISION        :: v             ! MSE (white noise) for node
  DOUBLE PRECISION        :: eps(1:10100)  ! the epsilons for the node
  DOUBLE PRECISION        :: eta(-99:10000)! the etas for the node
    TYPE (node), POINTER  :: reg, star     ! pointers to the regular and
  END TYPE node                            ! starred subnodes one level down

! These will contain the end results
```

```
  DOUBLE PRECISION        :: topphi(26), topv, acvf(1000)

! Other globals
  INTEGER                          :: n, m, method
  DOUBLE PRECISION, ALLOCATABLE :: x(:)
  CHARACTER                        :: stamp*4


CONTAINS

SUBROUTINE make_tree(original_lags)
  INTEGER, ALLOCATABLE       :: toplags(:)
  INTEGER, INTENT (IN)       :: original_lags(m)
  DOUBLE PRECISION           :: x(n)
  TYPE (node), POINTER       :: top_node

  NULLIFY (top_node)                  !associates top_node so we can use it
  ALLOCATE(toplags(m))
  toplags=original_lags

! now build the tree of node lags
  CALL build_node_tree(m,toplags,top_node)

! now fill the tree, ie. get coeffts and MSE's of each node
  CALL fill_tree(top_node)

! node tree built, so print it
  CALL print_node_tree(top_node)

! Likelihood calculation
  CALL likelihood(original_lags, topphi(1:m), topv)

  DEALLOCATE(toplags)

  RETURN
END SUBROUTINE make_tree

!*********************************************************************

RECURSIVE SUBROUTINE build_node_tree(lev,this_lags,this_node)
```

```
  INTEGER                  :: i, lev, this_lags(26)
  TYPE (node), POINTER     :: this_node

! This routine will create a tree of nodes needed to subset Burg model. The
! level and lags are assigned to each node. Also the MSE of each node is
! initialized to be zero so that later we'll be able to check which nodes
! have not yet been filled.

! first time thru' with a fresh node; point to it & make its J and J*
! pointers point nowhere
  IF (.NOT. ASSOCIATED(this_node)) THEN
    ALLOCATE (this_node)
    this_node%level=lev
    this_node%lags(1:lev)=this_lags(1:lev)
!   the check for an unfilled node will be that its MSE=0
    this_node%v=0
    NULLIFY (this_node%reg)
    NULLIFY (this_node%star)
  END IF

! recursive call to routine with J lags; only if level>1
!  lev=this_node%level
  IF (this_node%level>1) THEN
    this_lags(1:lev-1)=this_node%lags(1:lev-1)
    CALL build_node_tree(lev-1,this_lags,this_node%reg)
  END IF

! recursive call to routine with J* lags; only if level>1
!  lev=this_node%level
  IF (this_node%level>1) THEN
    this_lags(1:lev-1)=(/(this_node%lags(lev)              &
                                -this_node%lags(lev-i), i=1,lev-1)/)
    CALL build_node_tree(lev-1,this_lags,this_node%star)
  END IF

  RETURN
END SUBROUTINE build_node_tree

!***********************************************************************
```

```
RECURSIVE SUBROUTINE fill_tree(this_node)
  INTEGER               :: i
  TYPE (node), POINTER  :: this_node

! Here we fill in the coeffts and MSE for each node

  IF (this_node%level>1) THEN
    CALL fill_tree(this_node%reg)
!   if reg node has not been filled, then fill it!
    IF (this_node%reg%v==0) THEN
      CALL fill_node(this_node%reg)
    END IF

    CALL fill_tree(this_node%star)

!   if star node has not been filled, then fill it!
    IF (this_node%star%v==0) THEN
      CALL fill_node(this_node%star)
    END IF
  END IF

! now that whole tree is filled, we can fill top node
  IF (this_node%level==m) THEN
    CALL fill_node(this_node)
  END IF

  RETURN
END SUBROUTINE fill_tree

!*********************************************************************

SUBROUTINE fill_node(this_node)
  INTEGER                       :: i, km, lev, sum_range(2)
  DOUBLE PRECISION, ALLOCATABLE :: eps_J(:),eps_Js(:),eta_J(:),eta_Js(:)
  DOUBLE PRECISION              :: v_J, v_Js, phi_K, phi_Ks, top, sum_en
  DOUBLE PRECISION              :: phi_J(26), phi_Js(26), sum_e, sum_n
  TYPE (node), POINTER          :: this_node

! This is the routine where the real work of building the coeffts, MSEs,
! epsilons and etas is done
```

```
! initialize the precursors before applying algo to this node
  lev=this_node%level
  km=this_node%lags(lev)
  ALLOCATE (eps_J(1:n+km),eps_Js(1:n+km),eta_J(1-km:n),eta_Js(1-km:n))
  IF (lev==1) THEN
    eps_J =0.0
    eps_Js=0.0
    eta_J =0.0
    eta_Js=0.0
    eps_J (1:n)=x(1:n)
    eps_Js(1:n)=x(1:n)
    eta_J (1:n)=x(1:n)
    eta_Js(1:n)=x(1:n)
    v_J   =SUM(x**2)/n
    v_Js  =v_J
  ELSE  !we're at a higher level, so use reg and star node info
    phi_J (1:lev-1)=this_node%reg%phi (1:lev-1)
    phi_Js(1:lev-1)=this_node%star%phi(1:lev-1)
    eps_J  (1:n+km)=this_node%reg%eps (1:n+km)
    eps_Js (1:n+km)=this_node%star%eps(1:n+km)
    eta_J  (1-km:n)=this_node%reg%eta (1-km:n)
    eta_Js (1-km:n)=this_node%star%eta(1-km:n)
    v_J   =this_node%reg%v
    v_Js  =this_node%star%v
  END IF

! Initial conditions set, now apply algo to this node
! First the reflection coefficients
  IF (method==1) THEN ! YuWa
    sum_range=(/1,n+km/)
    top=0
    DO i=1,n+km
      top=top+(eps_J(i)*eta_Js(i-km))
    END DO
    phi_K =top/(n*v_Js)
    phi_Ks=top/(n*v_J)
  ELSE ! Burg type
    sum_range=(/1+km,n/)
!   First get sum of squares and cross squares for eps and eta:
```

```
    sum_e=0; sum_n=0; sum_en=0
    DO i=km+1,n
      sum_en=sum_en + eps_J(i)*eta_Js(i-km)
      sum_e=sum_e + eps_J(i)**2
      sum_n=sum_n + eta_Js(i-km)**2
    END DO
!   Now calculate reflection coefficients depending on the method:
    SELECT CASE (method)
      CASE (2)  ! Burg
        phi_K =v_J*(v_J+v_Js)*sum_en/(sum_n*v_J**2 + sum_e*v_Js**2)
        phi_Ks=v_Js*(v_J+v_Js)*sum_en/(sum_n*v_J**2 + sum_e*v_Js**2)
      CASE (3)  ! Morf
        phi_K =SQRT(v_J/(v_Js*sum_e*sum_n))*sum_en
        phi_Ks=SQRT(v_Js/(v_J*sum_e*sum_n))*sum_en
      CASE (4)  ! Nutt
        phi_K =2.0*v_J*sum_en/(v_Js*sum_e+v_J*sum_n)
        phi_Ks=v_Js*phi_K/v_J
    END SELECT
  END IF

! Continue with remaining recursions - identical for all algo's
  this_node%phi(lev)=phi_K
  IF (lev > 1) THEN
    DO i=1,lev-1
      this_node%phi(i)=phi_J(i)-phi_K*phi_Js(lev-i)
    END DO
  END IF
  this_node%v=(1-phi_Ks*phi_K)*v_J

! The eta's & epsilon's for posterity:
  DO i=sum_range(1),sum_range(2)
    this_node%eps(i)   =eps_J(i)-phi_K*eta_Js(i-km)
    this_node%eta(i-km)=eta_J(i-km)-phi_K*eps_Js(i)
  END DO

  DEALLOCATE (eps_J,eps_Js,eta_J,eta_Js)

  RETURN
END SUBROUTINE fill_node
```

```
!*********************************************************************

RECURSIVE SUBROUTINE print_node_tree(this_node)
  INTEGER               :: i
  TYPE (node), POINTER  :: this_node

! Here we print the info in the top node - can also make it print all nodes
! by removing the inmost IF THEN loop

  IF (ASSOCIATED(this_node)) THEN
    CALL print_node_tree(this_node%reg)
    IF (this_node%level==m) THEN
      PRINT*,"***************************************************"
      PRINT*,"The estimated subset ",stamp," AR coefficients are:"
      DO i=1,this_node%level
        PRINT 10, this_node%lags(i), this_node%phi(i)
 10     FORMAT(" Phi(",I2,"): ",F9.5)
      END DO
      PRINT*,"***************************************************"
      PRINT*, stamp," WN variance estimate : ",this_node%v
!     Store results for likelihood calcs
      topphi(1:m)=this_node%phi(1:m)
      topv=this_node%v
    END IF
    CALL print_node_tree(this_node%star)
    IF (this_node%level<m) DEALLOCATE (this_node)
  END IF
  RETURN
END SUBROUTINE print_node_tree

!*********************************************************************

SUBROUTINE likelihood(lag, phi, s2)
! First computes ACVF of a subset AR model with m coeffts (phi) and lags
! (lag), and sigma^2=s2, into acvf, lags 0 to n-1 acvf(0:n-1). Then it gets
! -2 log  Likelihood for the vector of obs x:
! -2log L = n*log(2*pi*s2) + sum(log(r(j))) + resid_ss/s2
! using the innnovations algorithm
  INTEGER    :: i, j, k, km, t
  INTEGER    :: lag(m), lags(m+1)
```

```
  DOUBLE PRECISION :: phi(m), phis(m+1), acvf(0:2*lag(m)), Ka(n,n)
  DOUBLE PRECISION :: A(lag(m)+1, lag(m)+1), b(lag(m)+1)
  DOUBLE PRECISION :: s, s2, pi, resid_ss, xh(n), loglike_wn, loglike_ss
  DOUBLE PRECISION :: cond_like, th(1:n-1,0:n-1), r(0:n-1), aicc

  km=lag(m)
  pi=3.141592654
  lags(1)    =0
  lags(2:m+1)=lag
  phis(1)    =-1.0
  phis(2:m+1)=phi
  b=0.0
  b(1)=s2
! Solve system A*acvf(0:km) = b, to get acvf(0:km)
  DO i=1,km+1
    A(i,:)=0.0
  END DO
  DO k=0,km
    DO j=0,m
      A(k+1, ABS(k-lags(j+1))+1) = A(k+1, ABS(k-lags(j+1))+1) - phis(j+1)
    END DO
  END DO
  CALL DLSARG(1+km, A, 1+km, b, 1, acvf(0:km))

! Now get acvf(km+1:2*km) via recursions
  DO k=km+1, 2*km
    acvf(k)=0.0
    DO j=1,m
      acvf(k)=acvf(k)+phi(j)*acvf(k-lag(j))
    END DO
  END DO

! Form K(.,.) as in (5.3.5)
  DO i=1,n
    Ka(i,:)=0.0
  END DO
  DO i=1,n
    DO j=1,n
      IF (i<=km .AND. j<=km) Ka(i,j)=acvf(abs(i-j))/s2
      IF (min(i,j)<=km .AND. km<max(i,j) .AND. max(i,j)<=2*km) THEN
```

```
        s=0
        DO k=1,m
          s=s+phi(k)*acvf(abs(lag(k)-abs(i-j)))
        END DO
        Ka(i,j)=(acvf(abs(i-j))-s)/s2
      END IF
      IF (min(i,j)>km .AND. i==j) Ka(i,j)=1.0
     END DO
  END DO

! Form (5.2.16) recursions
  th(:,0)=1.0
  r(0)=Ka(1,1)
  DO i=1,n-1
    DO k=0,i-1
      s=0
      DO j=0,k-1
        s=s+th(k,k-j)*th(i,i-j)*r(j)
      END DO
      th(i,i-k)=(Ka(i+1,k+1)-s)/r(k)
    END DO
    s=0
    DO j=0,i-1
      s=s+r(j)*th(i,i-j)**2
    END DO
    r(i)=Ka(i+1,i+1)-s
  END DO

! Build 1-step predictors (xh's), and get the resid_ss
  xh(1)=0.0
  DO k=1,n-1
    IF (k < km) THEN
      xh(k+1)=0.0
      DO j=1,k
        xh(k+1)=xh(k+1)+th(k,j)*(x(k+1-j)-xh(k+1-j))
      END DO
    ELSE ! k >= km
      xh(k+1)=0.0
      DO j=1,m
        xh(k+1)=xh(k+1)+phi(j)*x(k+1-lag(j))
```

```
      END DO
    END IF
  END DO
  s=0
  resid_ss=0
  DO j=1,n
    IF (r(j-1)<=0) THEN
      PRINT*, "### NON-CAUSAL MODEL ###"
      RETURN
    END IF
    s=s+log(r(j-1))
    resid_ss=resid_ss+(x(j)-xh(j))**2/r(j-1)
  END DO

! calculate cond. likelihood
! -2 log CL = (n-km)*log(s2)+(1/s2)sum_{km+1}^n (x_t-phi_k1*x_{t-k1}-...
!            -phi_km*x_{t-km})^2
  cond_like=0
  DO t=km+1,n
    cond_like=cond_like+DOT_PRODUCT(phi,(/(x(t-lag(j)), j=1,m)/))
  END DO
  cond_like=(n-m)*log(s2)+cond_like/s2

! Finally: -2log Likelihood=loglike: wn means use WN variance estimate:
! SS means use RSS/n variance estimate
  loglike_wn = n*log(2.0*pi*s2) + s + resid_ss/s2
  loglike_ss = n*log(2.0*pi*resid_ss/n) + s + FLOAT(n)
  PRINT *,"RSS/n WN variance estimate: ", resid_ss/n
  PRINT*,"-2 Log Like (",stamp,") :",loglike_wn
  PRINT*,"-2 Log Like (RSS/n):",loglike_ss
  aicc=loglike_ss+2.0*n*(m+1)/FLOAT(n-m-2)
  PRINT*,"AICC (RSS/n): ", aicc
!      PRINT*, " s ",s
!  PRINT *,"-2 Log Cond Like (YW WN): ", cond_like
  PRINT*,"%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%"

  RETURN
END SUBROUTINE likelihood

END MODULE tree
```

```
!**********************************************************************
!**********************************************************************

PROGRAM bdt

! Author: A. Trindade, www.stat.ufl.edu/~trindade/
! This program subset models AR(p)'s using Burg type recursions
! with subset size <=26, km<100, and a max of 10,000 observations.
! For details refer to the paper: "Implementing Modified Burg Algorithms
! in Multivariate Subset Autoregression", by the author.

  USE tree
  INTEGER                                 :: i, mc
  INTEGER, ALLOCATABLE, DIMENSION (:)     :: toplags
  DOUBLE PRECISION                        :: y(10000), mean
  CHARACTER                               :: h*24

  PRINT*, "%%%%%%%%%%%%%%%% Univariate SAR Modeling Program %%%%%%%%%%%%%%%%"
! read in the time series
  20   write(*,*)
       write(*,22)
  22    format(5x,'File name of time series for modelling: ',$)
  23    h='                         '
        read(*,*) h
        IF (h=='a                        ') h='lynx10.tsm'
        open(3,file=h,status='old',err=20)
        i=1
  25    read(3,*,end=30) y(i)
        i=i+1
          if (i.eq.10002) then
           write(*,6665)
6665       format(3x,'DATA TRUNCATED AFTER FIRST 10000 OBSERVATIONS.')
           i=i-1
           goto 30
          endif
        goto 25
  30    n=i-1
        close(3)
```

```
! now mean-correct the obs
  mean=SUM(y)/n
  ALLOCATE (x(n))
  x=(/(y(i),i=1,n)/)
  PRINT*, "Do you wish to mean-correct the observations (1=yes, 0=no)?"
  READ*, mc
  IF (mc==1) x=(/(y(i)-mean,i=1,n)/)
  PRINT*, "There are ",n," observations."
  PRINT*,"First obs is ",x(1)," last is ",x(n)

! Enter how many lags will be modeled
  PRINT*, "Enter the number of lags to be modeled (<27):"
  READ*, m
  ALLOCATE (toplags(m))

! read in the lags
  PRINT*, "Enter the lags:"
  READ*, (toplags(i), i=1,m)
! PRINT*,"The lags are: ", (toplags(i), i=1,m)

! Enter the method
  PRINT*, "Enter the method for obtaining the reflection coefficients:"
  PRINT*, "Yule-Walker (1), Burg (2), Vieira-Morf (3), Nuttall-Strand (4):"
  READ*, method
  SELECT CASE (method)
    CASE (1); stamp="YuWa"
    CASE (2); stamp="Burg"
    CASE (3); stamp="Morf"
    CASE (4); stamp="Nutt"
    CASE DEFAULT
      PRINT*,"Method not in range: ", method
      STOP
  END SELECT

  CALL make_tree(toplags)

END PROGRAM bdt
```

## C.2 Code for BDT2.F90

```
MODULE tree

  TYPE vector
    DOUBLE PRECISION         :: vec(2)
  END TYPE vector

  TYPE matrix
    DOUBLE PRECISION         :: mat(2,2)
  END TYPE matrix

! Here we define the data type NODE which will contain
  TYPE node
    INTEGER                  :: level         ! level in tree: top=m, bottom=1
    INTEGER                  :: lags(26)      ! lags for node are stored here
    TYPE (matrix)            :: A(26)         ! forward coefficients for node
    TYPE (matrix)            :: B(26)         ! backward coefficients for node
    TYPE (matrix)            :: vf, vb        ! forward and backward MSEs
    TYPE (vector)            :: eps(1:10100)  ! the epsilons for the node
    TYPE (vector)            :: eta(-99:10000)! the etas for the node
    TYPE (node), POINTER     :: reg, star     ! pointers to the regular and
  END TYPE node                               ! starred subnodes one level down

! These will contain the end results
    TYPE (matrix)            :: topA(1:26), topvf
! Other globals
  INTEGER                    :: n, m, method
  INTEGER, ALLOCATABLE       :: orig_lags(:)
  TYPE (vector), ALLOCATABLE :: x(:)
  CHARACTER                  :: stamp*4


CONTAINS

SUBROUTINE make_tree(toplags, truevf)
  INTEGER                    :: toplags(m)
  TYPE (node), POINTER       :: top_node
  DOUBLE PRECISION           :: truevf(3)
  EXTERNAL DEVLRG
```

```
  NULLIFY (top_node)                  !associates top_node so we can use it
  ALLOCATE(orig_lags(m))
  orig_lags=toplags

! now build the tree of node lags
  CALL build_node_tree(m,toplags,top_node)

! now fill the tree, i.e. get coeffts and MSE's of each node
  CALL fill_tree(top_node)

! node tree built, so print it
  CALL print_node_tree(top_node)

! undo node tree so that we don't run out of memory for next runs
! CALL undo_node_tree(top_node)

! Causal check - program will terminate if noncausal solution.
  CALL Causal_Check

! Likelihood calculations (exact or approx)
!  CALL likelihood("YW WN")
  CALL approx_likelihoods(truevf)

  RETURN
END SUBROUTINE make_tree

!*********************************************************************

RECURSIVE SUBROUTINE build_node_tree(lev,this_lags,this_node)
  INTEGER                 :: i, lev, this_lags(26)
  TYPE (node), POINTER    :: this_node

! This routine will create a tree of nodes needed to subset Burg model. The
! level and lags are assigned to each node. Also the MSE of each node is
! initialized to be zero so that later we'll be able to check which nodes
! have not yet been filled.

! first time thru' with a fresh node; point to it & make its J and J*
! pointers point nowhere
```

```
  IF (.NOT. ASSOCIATED(this_node)) THEN
    ALLOCATE (this_node)
    this_node%level=lev
    this_node%lags(1:lev)=this_lags(1:lev)
!   the check for an unfilled node will be the (1,1) entry of vf=0
    this_node%vf%mat(1,1)=0
    NULLIFY (this_node%reg)
    NULLIFY (this_node%star)
  END IF

! recursive call to routine with J lags; only if level>1
  lev=this_node%level
  IF (lev>1) THEN
    this_lags(1:lev-1)=this_node%lags(1:lev-1)
    CALL build_node_tree(lev-1,this_lags,this_node%reg)
  END IF

! recursive call to routine with J* lags; only if level>1
  lev=this_node%level
  IF (lev>1) THEN
    this_lags(1:lev-1)=(/(this_node%lags(lev)              &
                                -this_node%lags(lev-i), i=1,lev-1)/)
    CALL build_node_tree(lev-1,this_lags,this_node%star)
  END IF

  RETURN
END SUBROUTINE build_node_tree

!*********************************************************************

RECURSIVE SUBROUTINE fill_tree(this_node)
  INTEGER               :: i
  TYPE (node), POINTER  :: this_node

! Here we fill in the coeffts and MSE for each node

  IF (this_node%level>1) THEN
    CALL fill_tree(this_node%reg)
!   if reg node has not been filled, then fill it!
    IF (this_node%reg%vf%mat(1,1)==0) THEN
```

```
      CALL fill_node(this_node%reg)
    END IF

    CALL fill_tree(this_node%star)

!   if star node has not been filled, then fill it!
    IF (this_node%star%vf%mat(1,1)==0) THEN
      CALL fill_node(this_node%star)
    END IF
  END IF

! now that whole tree is filled, we can fill top node
  IF (this_node%level==m) THEN
    CALL fill_node(this_node)
  END IF

  RETURN
END SUBROUTINE fill_tree

!*********************************************************************

SUBROUTINE fill_node(this_node)
  INTEGER               :: i, j, t, km, lev, range(2)
  TYPE (vector), ALLOCATABLE       :: eps_J(:),eps_Js(:),eta_J(:),eta_Js(:)
  DOUBLE PRECISION, DIMENSION(2,2) :: vf_J, vf_Js, A_K, B_Ks, sen, see, Id
  DOUBLE PRECISION, DIMENSION(2,2) :: vfinv, vfsinv, B, A_Ks, B_K,vb_J,vb_Js
  DOUBLE PRECISION, DIMENSION(2,2) :: vbinv, vbsinv, snn, R, see2, snn2
  DOUBLE PRECISION, DIMENSION(2,2) :: U2, V2, AA, CC, tV
  DOUBLE PRECISION, DIMENSION(4,4) :: term1, term1_left, term1_right, A
  TYPE (matrix)         :: A_J(26), B_Js(26), A_Js(26), B_J(26)
  TYPE (node), POINTER  :: this_node

! This is the routine where the real work of building the coeffts, MSEs,
! epsilons and etas is done

! initialize the precursors before applying algo to this node
  lev=this_node%level
  km=this_node%lags(lev)
  ALLOCATE(eps_J(1:n+km),eps_Js(1:n+km),eta_J(1-km:n),eta_Js(1-km:n))
  IF (lev==1) THEN
```

```
    DO t=n+1,n+km
      eps_J(t)%vec =0; eps_Js(t)%vec=0
    END DO
    DO t=1-km,0
      eta_J(t)%vec =0; eta_Js(t)%vec=0
    END DO
    eps_J (1:n)=x(1:n)
    eps_Js(1:n)=x(1:n)
    eta_J (1:n)=x(1:n)
    eta_Js(1:n)=x(1:n)
    DO i=1,2
      DO j=1,2
         vf_J(i,j)=DOT_PRODUCT(x%vec(i),x%vec(j))/n
      END DO
    END DO
    vf_Js  =vf_J
    vb_Js  =vf_J
    vb_J  =vf_J
  ELSE  !we're at a higher level, so use reg and star node info
    A_J (1:lev-1)  =this_node%reg%A(1:lev-1)
    A_Js(1:lev-1)  =this_node%star%A(1:lev-1)
    B_Js(1:lev-1)  =this_node%star%B(1:lev-1)
    B_J (1:lev-1)  =this_node%reg%B(1:lev-1)
    eps_J (1:n+km) =this_node%reg%eps(1:n+km)
    eps_Js(1:n+km) =this_node%star%eps(1:n+km)
    eta_J (1-km:n) =this_node%reg%eta(1-km:n)
    eta_Js(1-km:n) =this_node%star%eta(1-km:n)
    vf_J  =this_node%reg%vf%mat
    vf_Js =this_node%star%vf%mat
    vb_Js =this_node%star%vb%mat
    vb_J  =this_node%reg%vb%mat
  END IF

! Initial conditions set, now apply algo to this node
! First build A_K(km) -------------------------------
  range=(/1,n+km/)
  IF (method>1) range=(/1+km,n/)
  DO i=1,2
    see(i,:)=(/0.0D+00,0.0D+00/)
    sen(i,:)=(/0.0D+00,0.0D+00/)
```

```
    snn(i,:)=(/0.0D+00,0.0D+00/)
  END DO
  DO t=range(1),range(2)
    DO i=1,2
      DO j=1,2
        see(i,j)=see(i,j)+eps_J(t)%vec(i)*eps_J(t)%vec(j)
        sen(i,j)=sen(i,j)+eps_J(t)%vec(i)*eta_Js(t-km)%vec(j)
        snn(i,j)=snn(i,j)+eta_Js(t-km)%vec(i)*eta_Js(t-km)%vec(j)
      END DO
    END DO
  END DO
! form I
  Id(1,:)=(/1.0D+00,0.0D+00/)
  Id(2,:)=(/0.0D+00,1.0D+00/)
! get the inverse of vf_J, put into vfinv
  CALL DLINRG(2,vf_J,2,vfinv,2)
  SELECT CASE (method)
    CASE (1)
!     get the inverse of vb_Js, put into vbsinv
      CALL DLINRG(2,vb_Js,2,vbsinv,2)
!     finally, get A_K(km) & B_Ks(km)
      A_K =MATMUL(sen, vbsinv)/FLOAT(n)
      B_Ks=MATMUL(TRANSPOSE(sen), vfinv)/FLOAT(n)
    CASE (2)
      B=sen+MATMUL(MATMUL(vfinv,sen),vb_Js)
      CALL KRON(2,snn,Id,term1_left)
      CALL KRON(2,MATMUL(vb_Js,vb_Js),MATMUL(MATMUL(vfinv,see),vfinv), &
       &      term1_right)
      term1=term1_left+term1_right
      CALL DLINRG(4,term1,4,A,4)
!     finally, build elements of A_K piecemeal
      DO i=1,2
        A_K(i,1)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
      DO i=3,4
        A_K(i-2,2)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
!     Finished building A_K ---------------- now get B_Ks from it
      B_Ks=MATMUL(MATMUL(vb_Js,TRANSPOSE(A_K)),vfinv)
    CASE(3)
```

```
      CALL Matrix_Power(vf_J, 5.0D-1, U2)
      CALL Matrix_Power(vb_Js, -5.0D-1, V2)
      CALL Matrix_Power(see, -5.0D-1, see2)
      CALL Matrix_Power(snn, -5.0D-1, snn2)
      R=MATMUL(MATMUL(see2,sen),snn2)
      A_K=MATMUL(MATMUL(U2,R),V2)
      B_Ks=MATMUL(MATMUL(vb_Js,TRANSPOSE(A_K)),vfinv)
    CASE(4)
      CALL DLINRG(2,vb_Js,2,V2,2) ! V2=(V_J*)^-1
      U2=vfinv                    ! U2=(U_J)^-1
      B=MATMUL(snn,V2)
      AA=MATMUL(see,U2)
      CALL KRON(2,Id,AA,term1_left)
      CALL KRON(2,B,Id,term1_right)
      term1=term1_left+term1_right
      CALL DLINRG(4,term1,4,A,4)
      CC=2.0*sen
!     Now vec(R) = A . vec(CC)
      B=CC
!     finally, build elements of R piecemeal
      DO i=1,2
        R(i,1)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
      DO i=3,4
        R(i-2,2)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
!     Finished building R --------------  now get A_K & B_Ks from it
      A_K=MATMUL(R,V2)
      B_Ks=MATMUL(MATMUL(vb_Js,TRANSPOSE(A_K)),vfinv)
  END SELECT

! now get A_Ks, just swap star and nostar
  DO i=1,2
    see(i,:)=(/0.0D+00,0.0D+00/)
    sen(i,:)=(/0.0D+00,0.0D+00/)
    snn(i,:)=(/0.0D+00,0.0D+00/)
  END DO
  DO t=range(1),range(2)
    DO i=1,2
      DO j=1,2
```

```
        see(i,j)=see(i,j)+eps_Js(t)%vec(i)*eps_Js(t)%vec(j)
        sen(i,j)=sen(i,j)+eps_Js(t)%vec(i)*eta_J(t-km)%vec(j)
        snn(i,j)=snn(i,j)+eta_J(t-km)%vec(i)*eta_J(t-km)%vec(j)
      END DO
    END DO
  END DO
! get the inverse of vf_Js, put into vfsinv
  CALL DLINRG(2,vf_Js,2,vfsinv,2)
  SELECT CASE (method)
    CASE (1)
!     get the inverse of vb_J, put into vbinv
      CALL DLINRG(2,vb_J,2,vbinv,2)
!     finally, get A_Ks(km) & B_K(km)
      A_Ks=MATMUL(sen, vbinv)/FLOAT(n)
      B_K =MATMUL(TRANSPOSE(sen), vfsinv)/FLOAT(n)
    CASE (2)
      B=sen+MATMUL(MATMUL(vfsinv,sen),vb_J)
      CALL KRON(2,snn,Id,term1_left)
      CALL KRON(2,MATMUL(vb_J,vb_J),MATMUL(MATMUL(vfsinv,see),vfsinv), &
        &     term1_right)
      term1=term1_left+term1_right
      CALL DLINRG(4,term1,4,A,4)
!     finally, build elements of A_Ks piecemeal
      DO i=1,2
        A_Ks(i,1)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
      DO i=3,4
        A_Ks(i-2,2)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
!     Finished building A_Ks ---------------
!     now get B_K from it
      B_K=MATMUL(MATMUL(vb_J,TRANSPOSE(A_Ks)),vfsinv)
    CASE(3)
      CALL Matrix_Power(vf_Js, 5.0D-1, U2)
      CALL Matrix_Power(vb_J, -5.0D-1, V2)
      CALL Matrix_Power(see, -5.0D-1, see2)
      CALL Matrix_Power(snn, -5.0D-1, snn2)
      R=MATMUL(MATMUL(see2,sen),snn2)
      A_Ks=MATMUL(MATMUL(U2,R),V2)
      B_K=MATMUL(MATMUL(vb_J,TRANSPOSE(A_Ks)),vfsinv)
```

```
    CASE(4)
      CALL DLINRG(2,vb_J,2,V2,2) ! V2=(V_J)^-1
      U2=vfsinv                  ! U2=(U_J*)^-1
      B=MATMUL(snn,V2)
      AA=MATMUL(see,U2)
      CALL KRON(2,Id,AA,term1_left)
      CALL KRON(2,B,Id,term1_right)
      term1=term1_left+term1_right
      CALL DLINRG(4,term1,4,A,4)
      CC=2.0*sen
!     Now vec(R) = A . vec(CC)
      B=CC
!     finally, build elements of R piecemeal
      DO i=1,2
        R(i,1)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
      DO i=3,4
        R(i-2,2)=A(i,1)*B(1,1)+A(i,2)*B(2,1)+A(i,3)*B(1,2)+A(i,4)*B(2,2)
      END DO
!     Finished building R --------------  now get A_Ks & B_K from it
      A_Ks=MATMUL(R,V2)
      B_K=MATMUL(MATMUL(vb_J,TRANSPOSE(A_Ks)),vfsinv)
  END SELECT


! assign A_K and B_K to correct node place
  this_node%A(lev)%mat=A_K
  this_node%B(lev)%mat=B_K
! Now do eqtns 1.16 and 1.18
  IF (lev > 1) THEN
    DO i=1,lev-1
      this_node%A(i)%mat=A_J(i)%mat-MATMUL(A_K,B_Js(lev-i)%mat)
      this_node%B(i)%mat=B_J(i)%mat-MATMUL(B_K,A_Js(lev-i)%mat)
    END DO
  END IF

! set MSE's, eqtns 1.19 and 1.20
  this_node%vf%mat=MATMUL((Id-MATMUL(A_K,B_Ks)),vf_J)
  this_node%vb%mat=MATMUL((Id-MATMUL(B_K,A_Ks)),vb_J)
```

```
! Set epsilons and etas
  DO i=range(1),range(2)
    this_node%eps(i)%vec   =eps_J(i)%vec-MATMUL(A_K,eta_Js(i-km)%vec)
    this_node%eta(i-km)%vec=eta_J(i-km)%vec-MATMUL(B_K,eps_Js(i)%vec)
  END DO

  DEALLOCATE(eps_J, eps_Js ,eta_J ,eta_Js)

  RETURN
END SUBROUTINE fill_node

!***********************************************************************

RECURSIVE SUBROUTINE print_node_tree(this_node)
  INTEGER               :: i, km, t, j
  TYPE (node), POINTER  :: this_node

! Here we print the info in the top node - can also make it print all nodes
! by removing the inmost IF THEN loop

  IF (ASSOCIATED(this_node)) THEN
    CALL print_node_tree(this_node%reg)
    IF (this_node%level==m) THEN
      PRINT*,"*************************************************"
      PRINT*,"Estimated subset ",stamp," coefficient matrices:"
      DO i=1,this_node%level
        PRINT*,"Phi(", this_node%lags(i),"):"
        CALL DWRRRL('',2,2,this_node%A(i)%mat,2,0, &
                   '(F20.6)','NONE','NONE')
      END DO
      PRINT*," "
      PRINT*,"*************************************************"
      PRINT*,"Estimated ",stamp," (forward) WN covariance matrix:"
      CALL DWRRRL('',2,2,this_node%vf%mat,2,0,'(F20.6)','NONE','NONE')
      PRINT*," "

!     Store results for likelihood calcs
      topA(1:m)=this_node%A(1:m)
      topvf=this_node%vf
    END IF
```

```
    CALL print_node_tree(this_node%star)
    IF (this_node%level<m) DEALLOCATE (this_node)
  END IF

  RETURN
END SUBROUTINE print_node_tree

!*********************************************************************

SUBROUTINE Matrix_Power(A, pow, B)
! Raises (2 by 2) matrix A to a real power pow, result into matrix B.

  INTEGER          :: ipath, irank
  DOUBLE PRECISION :: pow, tol, A(2,2), B(2,2), D(2,2), U(2,2), V(2,2), eig(2)

  ipath=11
  tol=1.0D-10

  CALL DLSVRR(2, 2, A, 2, ipath, tol, irank, eig, U, 2, V, 2)
  D=0.0
  D(1,1)=eig(1)**ABS(pow); D(2,2)=eig(2)**ABS(pow)

  B=MATMUL(MATMUL(U,D),TRANSPOSE(V))
  IF (pow<0) CALL DLINRG(2,B,2,B,2)

  RETURN
END SUBROUTINE Matrix_Power

!*********************************************************************

SUBROUTINE KRON (d,A,B,C)
! computes the Kronecker product of square matrices A and B (dim=d),
! puts into C (dim=d^2)
  INTEGER             :: d, i, j, k, l, row, col
  DOUBLE PRECISION    :: A(d,d), B(d,d), C(d**2,d**2)

  DO i=1,d
    DO k=1,d
      row=d*(i-1)+k
      DO j=1,d
```

```
        DO l=1,d
          col=d*(j-1)+l
          C(row,col)=A(i,j)*B(k,l)
        END DO
      END DO
    END DO
  END DO

  RETURN
END SUBROUTINE KRON

!*********************************************************************

SUBROUTINE Causal_Check
! Checks for causality of obtained solution. If noncausal, program will
! terminate without likelihood computations.

  INTEGER                       :: i, j, k, t, km
  INTEGER                       :: lag(m)
  TYPE (matrix)                 :: phi(m)
  DOUBLE PRECISION, ALLOCATABLE :: A(:,:)
  COMPLEX (KIND=8), ALLOCATABLE :: eig(:)
! kind=8 above specifies that eig's entries be double precision (LIB only).
  LOGICAL                       :: causal

  lag=orig_lags
  phi=topA(1:m)
  km=lag(m)
  ALLOCATE (A(2*km,2*km), eig(2*km))

! Build the matrix A ie. can write a VAR(p) as a VAR(1).
  A=0
  DO i=1,2*km
    IF (i<3) THEN
      DO j=1,m
        A(i,2*lag(j)-1)=phi(j)%mat(i,1)
        A(i,2*lag(j))  =phi(j)%mat(i,2)
      END DO
    ELSE ! i=>3
      A(i,i-2)=1.0
```

```
    END IF
  END DO

! For the Phi's to be causal, simply check that all 2*km eigenvalues
! of A are < 1 in absolute value.
  CALL DEVLRG(2*km, A, 2*km, eig)
  causal=.TRUE.
  DO i=1,2*km
    IF (ABS(eig(i)) >= 1.0) THEN !solution non-causal
      causal=.FALSE.
      EXIT
    END IF
  END DO

  IF (causal) THEN !if solution causal, return control to main prog.
    DEALLOCATE (A, eig)
    RETURN
  ELSE                          !solution non-causal, prog. will terminate.
    PRINT*,"%%% NON-CAUSAL SOLUTION. PROGRAM WILL TERMINATE. %%%"
    PRINT*,"************************************************"
    DEALLOCATE (A, eig)
    STOP
  END IF

END SUBROUTINE Causal_Check

!**********************************************************************

SUBROUTINE likelihood(what)
! Exact likelihood calculation: very, very slow...
! First computes ACVF of a 2d subset AR model with m coeffts (phi) and lags
! (lag), sigma^2=s2, into G(1-2km),...,G(-1),G(0),G(1),...,G(km-1).
! Then it gets -2 log  Likelihood for the vector of obs x: eqtn (11.5.5)
! in Yellow book, using the multivariate innnovations algorithm.

  INTEGER          :: i, j, k, t, km, drow
  INTEGER          :: lag(m)
  TYPE (vector)    :: xh(n)
  TYPE (matrix)    :: phi(m), s2, V(0:n-1), Vi(0:n-1)
  TYPE (matrix)    :: Ka(n,n), Th(n-1,n-1)
```

```
  DOUBLE PRECISION :: Z2(2,2), Tp(2,2), SumLogDetV, Term3, Like, temph(2)
  DOUBLE PRECISION :: tempv(2), DetV
  TYPE (matrix), ALLOCATABLE    :: G(:)
  DOUBLE PRECISION, ALLOCATABLE :: A(:,:), B(:,:), C(:,:), D(:,:)
  CHARACTER        :: what*7

  lag=orig_lags
  phi=topA(1:m)
  s2=topvf
  km=lag(m)
  pi=3.141592654
  Z2=0.0
  ALLOCATE (G(1-2*km:km-1),A(2*km,2*km),B(4*km**2,4*km**2), &
            C(4*km**2,4*km**2),D(4*km**2,4*km**2))

! Begin --- building the covariances Gamma(h)
! Build the matrix A
  A=0
  DO i=1,2*km
    IF (i<3) THEN
      DO j=1,m
        A(i,2*lag(j)-1)=phi(j)%mat(i,1)
        A(i,2*lag(j))  =phi(j)%mat(i,2)
      END DO
    ELSE ! i=>3
      A(i,i-2)=1.0
    END IF
  END DO
! B=A kronecker A
  CALL KRON(2*km,A,A,B)
! Form C = I - B
  C=-B
  DO i=1,4*km**2
    C(i,i)=1.0-B(i,i)
  END DO
! D = inv(C)
  CALL DLINRG(4*km**2, C, 4*km**2, D, 4*km**2)
! vec G_y(0) = D vec(s2). This forms G(1-km),...,G(-1),G(0),G(1),...,G(km-1)
  DO k=0,km-1
   DO i=1,2
```

```
    DO j=1,2
      drow=2*k+i+2*(j-1)*km
      G(-k)%mat(i,j)=D(drow,1)*s2%mat(1,1)+D(drow,2*km+2)*s2%mat(2,2) &
                    +(D(drow,2)+D(drow,2*km+1))*s2%mat(1,2)
    END DO
   END DO
   G(k)%mat=TRANSPOSE(G(-k)%mat)
  END DO
! Now compute G(-km),...,G(-(2km-1))
  DO k=km,2*km-1
    G(-k)%mat=0
    DO j=1,m
      G(-k)%mat=G(-k)%mat+MATMUL(G(lag(j)-k)%mat,TRANSPOSE(phi(j)%mat))
    END DO
  END DO
! print
!  DO k=1-2*km,2*km-1
!   PRINT*,"Gamma(",k,"):"
!   CALL DWRRRL('',2,2, G(k)%mat,2, 0,'(W20.6)','NONE','NONE')
!  END DO

! Begin --- building the K(i,j)'s, recursions (11.4.27)
  DO i=1,n
    DO j=1,n
      Ka(i,j)%mat=Z2
    END DO
  END DO
  DO i=1,n
    DO j=i,n
      IF (j<=km) Ka(i,j)%mat=G(i-j)%mat
      IF (i<=km .AND. km<j .AND. j<=2*km) THEN
        Ka(i,j)%mat=G(i-j)%mat
        DO k=1,m
          Ka(i,j)%mat=Ka(i,j)%mat-MATMUL(G(i-j+lag(k))%mat, &
                                         TRANSPOSE(phi(k)%mat))
        END DO
      END IF
      IF (i==j .AND. i>km) Ka(i,j)%mat=s2%mat
    END DO
  END DO
```

```
  DO j=1,n
    DO i=j+1,n
      Ka(i,j)%mat=TRANSPOSE(Ka(j,i)%mat)
    END DO
  END DO

! Begin --- building the theta(i,j)'s & V's, recursions (11.4.23)
  V(0)%mat=Ka(1,1)%mat
! Store inverses of V, the Vi's
  CALL DLINRG(2, V(0)%mat, 2, Vi(0)%mat, 2)
! Initialize & Keep total of sum of log(det(V))'s
  DetV=V(0)%mat(1,1)*V(0)%mat(2,2)-V(0)%mat(2,1)*V(0)%mat(1,2)
  SumLogDetV=log(DetV)
  DO t=1,n-1
    DO k=0,t-1
      Tp=0
      DO j=0,k-1
        Tp=Tp+MATMUL(MATMUL(Th(t,t-j)%mat,V(j)%mat),TRANSPOSE(Th(k,k-j)%mat))
      END DO
      Th(t,t-k)%mat=MATMUL((Ka(t+1,k+1)%mat-Tp),Vi(k)%mat)
    END DO
    Tp=0
    DO j=0,t-1
      Tp=Tp+MATMUL(MATMUL(Th(t,t-j)%mat,V(j)%mat),TRANSPOSE(Th(t,t-j)%mat))
    END DO
!   Get V's, their inverses Vi's, and keep total of sum of log((det(V))'s
    V(t)%mat=Ka(t+1,t+1)%mat - Tp
    CALL DLINRG(2, V(t)%mat, 2, Vi(t)%mat, 2)
    DetV=V(t)%mat(1,1)*V(t)%mat(2,2)-V(t)%mat(2,1)*V(t)%mat(1,2)
    SumLogDetV=SumLogDetV+log(DetV)
  END DO

! Get the one step predictors xh's, (11.4.28)
  xh(1)%vec=0
  Term3=DOT_PRODUCT(x(1)%vec,MATMUL(Vi(0)%mat,x(1)%vec))
  DO t=1,n-1
    xh(t+1)%vec=0
    IF (t<km) THEN
     DO j=1,t
      xh(t+1)%vec=xh(t+1)%vec+MATMUL(Th(t,j)%mat,x(t+1-j)%vec-xh(t+1-j)%vec)
```

```
     END DO
    ELSE ! t=>km
     DO j=1,m
      xh(t+1)%vec=xh(t+1)%vec+MATMUL(phi(j)%mat,x(t+1-lag(j))%vec)
     END DO
    END IF
!   Keep total of Term3 = sum_{t=1}^n [(x(t)-xh(t))'Vi(t-1)(x(t)-xh(t))]
    Term3=Term3+DOT_PRODUCT(x(t+1)%vec-xh(t+1)%vec,  &
                            MATMUL(Vi(t)%mat,x(t+1)%vec-xh(t+1)%vec))
  END DO

! Finally: -2 log Likelihood = Like
  Like=2.0*n*log(2.0*pi)+SumLogDetV+Term3

  DO t=0,km-1
    PRINT*,"Gamma(",t,"):"
    CALL DWRRRL('',2,2,G(t)%mat,2,0,'(F20.6)','NONE','NONE')
  END DO

  PRINT*,"-2 Log Like (",what,") : ", Like
!  PRINT*,"AICC (Burg WN):        ", Like+4.0*n*(4.0*km+1)/(2.0*(n-1)-4.0*km)

  DEALLOCATE (G,A,B,C,D)

  RETURN
END SUBROUTINE likelihood

!***********************************************************************

SUBROUTINE approx_likelihoods(truevf)
! Computes approx -2 log likelihood for the algorithm obtained
! Phi's (topA), and WN variance (topvf).
! Also computes WN variance estimate (RSS/n) starting with topvf WN estimate,
! that minimizes the -2 log likelihood for the algorithm obtained Phi's.

  INTEGER          :: i, maxitn
  DOUBLE PRECISION :: ff, tzz(3), step, oldff, truevf(3)
  DOUBLE PRECISION :: like1, like2, like

! Get likelihood for the WN variance estimate
```

```
  tzz(1)=topvf%mat(1,1); tzz(2)=topvf%mat(2,2); tzz(3)=topvf%mat(1,2)
  CALL lad_fun(3, tzz, oldff)
!  PRINT*,"AICC (Burg WN):        ", ff+4.0*n*(4.0*km+1)/(2.0*(n-1)-4.0*km)
  PRINT*,"-2 Log Like (",stamp,"): ", oldff

! Now Get WN estimate that maximizes likelihood, for the algorithm Phi's.
! This is equivalent to RSS/n in one dimension. First start search with
! topvf estimate from algo
  step=0.1
  CALL Hooke(3, tzz, step, ff)
  DO WHILE (ABS(ff-oldff) > 0.0001)
    oldff=ff
    step=step/10.0
    CALL Hooke(3, tzz, step, ff)
  END DO
  like1=MIN(ff, oldff)

! then use truevf (if it exists)
  IF (truevf(1)>0) THEN
    CALL lad_fun(3, truevf, oldff)
    step=0.1
    CALL Hooke(3, truevf, step, ff)
    DO WHILE (ABS(ff-oldff) > 0.0001)
      oldff=ff
      step=step/10.0
      CALL Hooke(3, truevf, step, ff)
    END DO
    like2=MIN(ff, oldff)
  ELSE
    like2=1.0D30
  END IF

! Now take lowest like as the RSS/n
  IF (like1 > like2) THEN
    like=like2
    tzz=truevf
  ELSE ! like1 smaller
    like=like1
  END IF
  PRINT*,"-2 Log Like (RSS/n): ", like
```

```
! get the RSS/n matrix
  topvf%mat(1,1)=tzz(1); topvf%mat(2,2)=tzz(2)
  topvf%mat(1,2)=tzz(3); topvf%mat(2,1)=tzz(3)
  PRINT*," "
  PRINT*,"Estimated RSS/n (forward) WN covariance matrix:"
  CALL DWRRRL('',2,2,topvf%mat,2,0,'(F20.6)','NONE','NONE')
  PRINT*,"%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%"

  RETURN
END SUBROUTINE approx_likelihoods

!*********************************************************************

SUBROUTINE lad_fun(nn, zz, ff)
! *** Objective function to go with H&J routine  ***
! Computes approx -2 log likelihood for the given Phi's (phi), and the 3
! components of the WN variance (zz). Uses recursions 11.3.12 (YB) to find
! Psi's. G(h) found by truncated summation h=0,...,l (instead of infty).

  INTEGER          :: i, j, k, t, km, l, h, nn
  INTEGER          :: lag(m)
  TYPE (matrix)    :: phi(m), s2, s2i
  DOUBLE PRECISION :: t1, t2, t3, t4, tvec(2), zz(*), ff
  DOUBLE PRECISION, ALLOCATABLE    :: GK(:,:), GKi(:,:), y(:)
  TYPE (matrix),    ALLOCATABLE    :: G(:), Psi(:), Phy(:)

  l=100          ! the truncation for the ACVF's from the PSi's
  lag=orig_lags
  phi=topA(1:m)
  km=lag(m)
  pi=3.141592654

! Build in constraints for +ve def. WN:
  IF ((zz(1)<0).OR.(zz(2)<0).OR.(zz(1)*zz(2)<zz(3)**2)) THEN
    ff=1.0D30
    RETURN
  END IF

!  IF (zz(1)<0.0) zz(1)=ABS(zz(1))
```

```
!  IF (zz(2)<0.0) zz(2)=ABS(zz(2))
!  IF (zz(1)*zz(2)<zz(3)**2) zz(3)=SQRT(zz(1)*zz(2))/2.0

! Build the WN, s2
  s2%mat(1,1)=zz(1); s2%mat(2,2)=zz(2)
  s2%mat(2,1)=zz(3); s2%mat(1,2)=zz(3)

  ALLOCATE (G(0:km-1), GK(2*km,2*km), Psi(0:km-1+l), Phy(1:km))
  ALLOCATE (y(2*km), GKi(2*km,2*km))

! Need to form Phy=0 except at lag(1:m)
  DO h=1,km
    Phy(h)%mat(1,:)=(/0.0,0.0/); Phy(h)%mat(2,:)=(/0.0,0.0/)
  END DO
  DO t=1,m
    Phy(lag(t))%mat=phi(t)%mat
  END DO

! Now get the Psi's
  Psi(0)%mat(1,:)=(/1.0,0.0/); Psi(0)%mat(2,:)=(/0.0,1.0/)
  DO j=1,km-1+l
    Psi(j)%mat(1,:)=(/0.0,0.0/); Psi(j)%mat(2,:)=(/0.0,0.0/)
    DO t=1,MIN(j,km)
      Psi(j)%mat=Psi(j)%mat+MATMUL(Phy(t)%mat,Psi(j-t)%mat)
    END DO
  END DO

! Now compute covariance matrices
  DO h=0,km-1
    G(h)%mat(1,:)=(/0.0,0.0/); G(h)%mat(2,:)=(/0.0,0.0/)
    DO j=0,l
      G(h)%mat=G(h)%mat+MATMUL(MATMUL(Psi(h+j)%mat,s2%mat), &
                                              TRANSPOSE(Psi(j)%mat))
    END DO
!    PRINT*,"Truncated Gamma(",h,"), with l=",l,":"
!    CALL DWRRRL('',2,2,G(h)%mat,2,0,'(F20.6)','NONE','NONE')
  END DO

! Form Big covariance matrix GK (symmetric):
  DO i=1,km
```

```
    DO j=1,i
      DO h=0,1
        DO k=0,1
          GK(2*i-h,2*j-k)=G(i-j)%mat(2-h,2-k)
          GK(2*j-k,2*i-h)=GK(2*i-h,2*j-k)
        END DO
      END DO
    END DO
  END DO

! Form term4
  t4=0.0
  CALL DLINDS(2, s2%mat, 2, s2i%mat, 2)
  DO t=1+km,n
    tvec=0.0
    DO k=1,m
      tvec=tvec+MATMUL(phi(k)%mat,x(t-lag(k))%vec)
    END DO
    tvec=x(t)%vec-tvec
    t4=t4+DOT_PRODUCT(tvec,MATMUL(s2i%mat,tvec))
  END DO

! Form t3
  CALL DLINDS(2*km, GK, 2*km, GKi, 2*km)
  DO j=1,km
    DO k=0,1
      y(2*j-k)=x(j)%vec(2-k)
    END DO
  END DO
  t3=DOT_PRODUCT(y,MATMUL(GKi,y))

! Form t2
  t2=(n-km)*LOG(s2%mat(1,1)*s2%mat(2,2)-s2%mat(1,2)**2)

! Form t1
! find det of GK by Choleski decomposing GK->GKi (re-use to save space),
! then det(GK)=(product of diagonal elements of GKi)**2.
  CALL DLFTDS(2*km,GK,2*km,GKi,2*km)
  t1=0.0
  DO h=1,2*km
```

```
    t1=t1+LOG(GKi(h,h))
  END DO
  t1=2*t1

! Now put it all together
  ff=2.0*n*log(2.0*pi)+t1+t2+t3+t4

  DEALLOCATE (G, GK, GKi, Psi, Phy, y)

  RETURN
END SUBROUTINE lad_fun

!**********************************************************************
!subroutine optimization using Hooke and Jeeves
SUBROUTINE Hooke(p,AR,er,fv)

        INTEGER p
        DOUBLE PRECISION AR(p),er,fv

        INTEGER flg,fi,ik,m
        DOUBLE PRECISION x1(p),x2(p),c(p),bx(p)
        DOUBLE PRECISION ac, min,bmin
        intent(inout) :: AR,er

                m=p
                ac=.4*er
                fi=0
                ik=0
                ij=0
                iq=1
                x1=AR

 3073   do 3075 i=1,m
 3075   c(i)=x1(i)
! 3080   if(iq.eq.0)then
!        write(*,*) '      <Computing Gaussian likelihood>'
!        endif
        call lad_fun(p,c,fv)
        min=fv
 3085   if(iq.eq.0)goto 3640
```

```
       goto 3600
!
!      Exploratory moves
!
 3100  do 3105 j=1,m
 3105  x2(j)=x1(j)
       do 3160 i=1,m
       do 3115 j=1,m
 3115  c(j)=x2(j)
       c(i)=x1(i)+er
       call lad_fun(p,c,fv)
 3120  if(fv.lt.min) then
          min=fv
          x2(i)=c(i)
       endif
       c(i)=x1(i)-er
       call lad_fun(p,c,fv)
 3130  if(fv.lt.min) then
          min=fv
          x2(i)=c(i)
       endif
 3160  continue
       flg=1
       do 3180 i=1,m
       if(x1(i).ne.x2(i))flg=0
 3180  continue
       if(flg.eq.0)goto 3195
               er=er/2
       if(er.lt.ac)fi=1
       if(ik.eq.0)goto 3285
       if(ik.eq.1)goto 3605
       goto 3100
 3195  if(ik.eq.0)goto 3285
       if(ik.eq.1)goto 3605
!
!      Pattern moves
!
 3200  do 3230 i=1,m
 3230  x1(i)=2*x2(i)-x1(i)
       do 3260 i=1,m
```

```
 3260   bx(i)=x2(i)
        !bvar=var
        bmin=min
        do 3276 i=1,m
 3276   c(i)=x1(i)
        call lad_fun(p,c,fv)
 3280   min=fv
        ik=0
        goto 3100
 3285   if(fi.eq.1)goto 3615
        if(min.ge.bmin)goto 3310
        goto 3200
 3310   do 3330 i=1,m
 3330   x1(i)=bx(i)
        !var=bvar
        min=bmin
        goto 3615
!
!       End of moves
!
 3600   ik=1
        goto 3100
 3605   if(fi.eq.1)goto 3640
 3610   goto 3200
 3615   if(fi.eq.1)goto 3640
        goto 3600
 3640   continue
!3680   fi=0
! 3687   d=(.5/sqrt(xn))/(5**iii)
 3647    continue
        if(iq.ne.0)then
         min=bmin
 4310    do 4330 i=1,p
           IF (i .le. p) AR(i)=bx(i)
 4330    x1(i)=bx(i)
         fv=min
        endif

        END SUBROUTINE Hooke
```

```
!**********************************************************************



END MODULE tree

!**********************************************************************
!**********************************************************************

PROGRAM Bdt2

! this program subset models AR(p)'s for 2-dimensional time series, using
! Burg type recursions, with subset size <=26, km<100, and a max of 10000 obs.

  USE tree
  INTEGER                                      :: i, wn_known, mc
  INTEGER, ALLOCATABLE, DIMENSION (:)          :: toplags
  TYPE (vector)                                :: y(10000)
  CHARACTER                                    :: h*24
  DOUBLE PRECISION                             :: mean1, mean2, truevf(3)

  PRINT*, "%%%%%%%%%%%%%% Bivariate SVAR Modeling Program %%%%%%%%%%%%%%"
! read in the time series from a data file
  20   write(*,*)
       write(*,22)
  22    format(5x,'Enter file name of time series for modelling: ',$)
  23    h='                         '
        read(*,*) h
        IF (h=='a                        ') h='sun2.tsm'
        open(3,file=h,status='old',err=20)
        i=1
  25    read(3,*,end=30) y(i)%vec(1), y(i)%vec(2)
        i=i+1
          if (i.eq.10002) then
           write(*,6665)
6665       format(3x,'DATA TRUNCATED AFTER FIRST 10000 OBSERVATIONS.')
           i=i-1
           goto 30
          endif
        goto 25
```

```
  30    n=i-1
        close(3)

! now mean-correct the obs
  mean1=SUM(y%vec(1))/n
  mean2=SUM(y%vec(2))/n
  ALLOCATE (x(n))
  x%vec(1)=(/(y(i)%vec(1), i=1,n)/)
  x%vec(2)=(/(y(i)%vec(2), i=1,n)/)
  PRINT*, "Do you wish to mean-correct the observations (1=yes, 0=no)?"
  READ*, mc
  IF (mc==1) THEN
    x%vec(1)=(/(y(i)%vec(1)-mean1, i=1,n)/)
    x%vec(2)=(/(y(i)%vec(2)-mean2, i=1,n)/)
  END IF
  PRINT*, "There are ",n," observations."
  WRITE(*,40) x(1), x(n)
  40   FORMAT("Firs obs is ",F20.4," last is ",F20.4)

! Enter how many lags will be modeled
  PRINT*, "Enter the number of lags to be modeled (<27):"
  READ*, m
  ALLOCATE (toplags(m))

! read in the lags
  PRINT*, "Enter the lags:"
  READ*, (toplags(i), i=1,m)
! PRINT*,"The lags are: ", (toplags(i), i=1,m)

! Simulation?
  PRINT*, "Is the true white noise covariance matrix known (1=yes, 0=no)?"
  READ*, wn_known
  IF (wn_known==1) THEN
    PRINT*, "Enter Sigma(1,1), Sigma(2,2), Sigma(1,2):"
    READ*, (truevf(i), i=1,3)
  ELSE
! don't know true WN, so set truevf(1)=0 as flag for routine approx_likelihoods
    truevf(1)=0
  END IF
```

```
! method
  PRINT*, "Enter the method for obtaining the reflection coefficients."
  PRINT*, "Yule-Walker (1), Burg (2), Vieira-Morf (3), Nuttal-Strand (4):"
  READ*, method
  SELECT CASE (method)
    CASE (1); stamp="YuWa"
    CASE (2); stamp="Burg"
    CASE (3); stamp="Morf"
    CASE (4); stamp="Nutt"
    CASE DEFAULT
      PRINT*,"Method not in range: ", method
      STOP
  END SELECT

  CALL make_tree(toplags, truevf)

END PROGRAM Bdt2
```