



## A Computer Program to Calculate Two-Stage Short-Run Control Chart Factors for $(X, MR)$ Charts

Matthew E. Elam  
The University of Alabama

Kenneth E. Case  
Oklahoma State University

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### Abstract

This paper is the second in a series of two papers that fully develops two-stage short-run  $(X, MR)$  control charts. This paper describes the development and execution of a computer program that accurately calculates first- and second-stage short-run control chart factors for  $(X, MR)$  charts using the equations derived in the first paper. The software used is **Mathcad**. The program accepts values for number of subgroups,  $\alpha$  for the  $X$  chart, and  $\alpha$  for the  $MR$  chart both above the upper control limit and below the lower control limit. Tables are generated for specific values of these inputs and the implications of the results are discussed. A numerical example illustrates the use of the program.

*Keywords:* control chart, short-run statistical process control, two-stage control charting, probability integral of the range, probability integral of the studentized range, distribution of the mean moving range.

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## 1. Introduction

Hillier's two-stage short-run theory of control charting has been applied to  $(\bar{X}, R)$  charts (see Hillier 1969),  $(\bar{X}, v)$  and  $(\bar{X}, \sqrt{v})$  charts (see Yang and Hillier 1970),  $(\bar{X}, s)$  charts (see Elam and Case 2005a), and  $(X, MR)$  charts (see Elam and Case 2006), where  $R$ ,  $v$ ,  $s$ , and  $MR$  are the range, variance, standard deviation, and moving range (size two) of a subgroup, respectively. Elam and Case (2001) and Elam and Case (2003) develop computer programs to address issues concerning the limited tabulated results provided by Hillier (1969) and Yang and Hillier (1970), respectively. Elam and Case (2005b) develop a computer program using the equations in Elam and Case (2005a) to calculate the factors required to determine two-stage short-run control limits for  $(\bar{X}, s)$  charts so that the application of these control charts

would not be limited.

This paper describes the development and execution of a computer program that uses the equations in [Elam and Case \(2006\)](#) to accurately calculate first- and second-stage short-run control chart factors for  $(X, MR)$  charts regardless of the number of subgroups,  $\alpha$  for the  $X$  chart, and  $\alpha$  for the  $MR$  chart both above the upper control limit and below the lower control limit ( $\alpha$  is the probability of a false alarm). The software used is **Mathcad** ([Mathsoft 2005](#)). The computer program also uses functions provided by the software.

This paper first presents the computer program. Tables generated by the program are then presented and compared with legitimate results in the literature. Also, implications of the tabulated results are discussed. Following a numerical example that illustrates the use of the program, final conclusions describing the contributions of the program are given. It should be noted that results from the program are for processes generating parts with independent measurements that follow a Normal distribution.

## 2. The computer program

This section of the paper presents the computer program, which is available at <http://www.jstatsoft.org/v15/i11/> or starting at <http://program.20m.com/>. The program has seven pages, each of which is further divided into sections. When executing the program, it is possible for a section of code in the program to turn red and have the error message “Unknown Error.” To correct this, delete one character in the red code and type it back in. Click the mouse arrow outside of the code. The code should turn black, indicating that the error has been eliminated. If not, repeat the procedure (it will eventually correct the problem).

### 2.1. Page 1

The first page of the program begins with the data entry section. The program requires the user to enter the following values: **alphaInd** ( $\alpha$  for the  $X$  chart), **alphaMRUCL** ( $\alpha$  for the  $MR$  chart above the UCL), **alphaMRLCL** ( $\alpha$  for the  $MR$  chart below the LCL), and **m** (number of subgroups (i.e., the number of  $MR$ s plus one)). If no lower control limit on the  $MR$  chart is desired, the entry for **alphaMRLCL** should be left blank (do not enter zero). Before a value can be entered, the cursor must be moved to the right side of the appropriate equal sign. This may be done using the arrow keys on the keyboard or by moving the mouse arrow to the right side of the equal sign and clicking once with the left mouse button. The program is activated by paging down once the last entry is made. **Mathcad** allows the user to immediately page down to the output section of the program (explained later in subsection 2.7) after the last entry is made.

The next part of page 1 is section 1.1 of the program. The value **TOL** is the tolerance. The calculations that use this value will be accurate to ten places to the right of the decimal. The functions **dnorm(x, 0, 1)** and **pnorm(x, 0, 1)** in **Mathcad** are the pdf and cdf, respectively, of the standard Normal distribution. The equations for the pdf and cdf are also given in case the **dnorm** or **pnorm** function fails to calculate a result. In **Mathcad**, an equation turns red if it does not calculate a result due to an error. If the **dnorm** function gives an error, type **f(x)** on the left side of the equal sign in Equation 1:

$$= \left( (2 \times \pi)^{-0.5} \right) \times \exp(-x^2/2) \quad (1)$$

If the `pnorm` function gives an error, type  $F(x)$  on the left side of the equal sign in Equation 2:

$$= \int_0^x f(t) dt \quad (2)$$

The equation for  $P(W)$ , the probability integral (or cumulative distribution function (cdf)) of the range for subgroups of size two sampled from a standard Normal population, is Equation 1 in [Elam and Case \(2006\)](#). As stated in [Elam and Case \(2006\)](#),  $W$  represents the (standardized) range  $w/\sigma$ , where  $w$  is the range of a subgroup and  $\sigma$  is the population standard deviation. The equation for `d2`, the mean of the distribution of the range  $W = w/\sigma$  for subgroups of size two sampled from a Normal population with mean  $\mu$  and variance equal to one, is Equation 2 in [Elam and Case \(2006\)](#).

## 2.2. Page 2

Page 2 of the program begins with section 2.1. The code in this section determines `wD4` and `wD3`, the `1-alphaMRUCL` and `alphaMRLCL` percentage points, respectively, of the distribution of the range  $W = w/\sigma$  for subgroup size two. The values `wD4` and `wD3` are used to determine the  $\alpha$ -based conventional upper and lower control chart constants, respectively, for the *MR* chart. The roots of the equations `DUCL(W)` and `DLCL(W)` are `wD4` and `wD3`, respectively, and are determined using `root` (a function in **Mathcad** that uses Ridder's or Brent's method to find the roots of an equation when root bracketing is used). The subprograms `Wseed1` and `Wseed2` generate seed values `seedD4` and `seedD3`, respectively, for Ridder's or Brent's method.

The subprogram `Wseed1` works as follows. Initially,  $W_0$  and  $W_1$  are set equal to 0.01 and 0.02, respectively.  $A_0$  and  $A_1$  result from evaluating `DUCL(W)` at  $W_0$  and  $W_1$ , respectively. The `while` loop begins by checking if the product of  $A_0$  and  $A_1$  is negative. If so, the root for `DUCL(W)` lies between 0.01 and 0.02. If not,  $W_0$  and  $W_1$  are incremented by 0.01.  $A_0$  and  $A_1$  are recalculated and if their product is negative, the root for `DUCL(W)` lies between 0.02 and 0.03. Otherwise, the `while` loop repeats. Once a root for `DUCL(W)` is bracketed, the bracketing values are passed out of the subprogram into the  $2 \times 1$  vector `seedD4` to be used by Ridder's or Brent's method to determine `wD4`. The subprogram `Wseed2` works similarly to construct the  $2 \times 1$  vector `seedD3` to be used by Ridder's or Brent's method to determine `wD3`, except the starting value is 0.001.

The next part of page 2 is section 2.2 of the program. The following is a description of the equations in section 2.2 of the program and their corresponding equation numbers in [Elam and Case \(2006\)](#):

- `d(x)`: the equation for determining  $\nu$ , Equation 5
- `h(x)`: the ratio of the variance to the squared mean, both of the  $\chi$  distribution with  $x$  degrees of freedom, Equation 6 (`h(x)` has been updated for **Mathcad**, hence “`lnGamma`” replacing “`gammaln`”)
- `r`: the ratio of the variance to the squared mean, both of the distribution of the mean moving range  $\bar{MR}/\sigma$ , Equation 7a
- `b` and `c`: Equations 7b and 7c, respectively

The values  $\mathbf{rprevm}$  and  $\mathbf{dprevm}(\mathbf{x})$  have the same meanings as  $\mathbf{r}$  and  $\mathbf{d}(\mathbf{x})$ , respectively, except they are for  $m - 1$  subgroups. The value  $\nu$  is the root of the equation  $\mathbf{d}(\mathbf{x})$  and is determined using `root` with seed value `seed $\nu$` . The value  $\nu\mathbf{prevm}$  is the root of the equation  $\mathbf{dprevm}(\mathbf{x})$  and is determined using `root` with seed value `seed $\nu\mathbf{prevm}$` . The subprogram `dfseed` generates the seed values `seed $\nu$`  and `seed $\nu\mathbf{prevm}$`  for Ridder's or Brent's method.

The subprogram `dfseed` works as follows. Initially,  $df_0$  and  $df_1$  are set equal to 0.9 and 1.1, respectively.  $A_0$  and  $A_1$  result from evaluating  $\mathbf{y}(\mathbf{x})$  (which is equal to either  $\mathbf{d}(\mathbf{x})$  or  $\mathbf{dprevm}(\mathbf{x})$ ) at  $df_0$  and  $df_1$ , respectively. The `while` loop begins by checking if the product of  $A_0$  and  $A_1$  is negative. If so, the root for  $\mathbf{y}(\mathbf{x})$  lies between 0.9 and 1.1. If not,  $df_0$  and  $df_1$  are incremented by 0.5.  $A_0$  and  $A_1$  are recalculated and if their product is negative, the root for  $\mathbf{y}(\mathbf{x})$  lies between 1.1 and 1.6. Otherwise, the `while` loop repeats. Once a root for  $\mathbf{y}(\mathbf{x})$  is bracketed, the bracketing values are passed out of the subprogram into the  $2 \times 1$  vector `seed $\nu$`  (if  $\mathbf{y}(\mathbf{x})$  is equal to  $\mathbf{d}(\mathbf{x})$ ) or `seed $\nu\mathbf{prevm}$`  (if  $\mathbf{y}(\mathbf{x})$  is equal to  $\mathbf{dprevm}(\mathbf{x})$ ) to be used by Ridder's or Brent's method to determine  $\nu$  or  $\nu\mathbf{prevm}$ , respectively. Both  $\nu$  and  $\nu\mathbf{prevm}$  are used to determine the two-stage short-run control chart factors for the  $(X, MR)$  charts.

### 2.3. Page 3

Page 3 of the program begins with section 3.1. The following is a description of the equations in section 3.1 of the program and their corresponding equation numbers in [Elam and Case \(2006\)](#):

- $P3(\mathbf{z})$ , the probability integral of the studentized range for subgroups of size two sampled from a Normal population, Equation 3a
- $c\nu$ ,  $P1(\mathbf{z})$ , and  $P2(\mathbf{z})$ , components of  $P3(\mathbf{z})$ , Equations 3b, 3c, and 3d, respectively ( $c\nu$  has been updated for **Mathcad**, hence “`ln $\Gamma$` ” replacing “`gammln`”)

As stated in [Elam and Case \(2006\)](#), the variable  $\mathbf{z}$  is equal to  $5 \times Q$ .  $Q$  represents the studentized range  $w/s$ , where  $w$  is the range of a subgroup and  $s$  is an independent estimate (based on  $\nu$  degrees of freedom) of the population standard deviation. The function `ln $\Gamma$`  in the equation for  $c\nu$  calculates the natural logarithm of the gamma ( $\Gamma$ ) function.

Section 3.2 contains the calculations required to determine `qD4`, the `1-alphaMRUCL` percentage point of the distribution of the studentized range  $Q = w/s$  for subgroup size two with  $\nu$  degrees of freedom. The value `qD4` is used to determine the second-stage short-run upper control chart factor for the  $MR$  chart. The subprogram `Zseed1` generates the seed value `seed1` for Ridder's or Brent's method (the first `root` function) or for the Secant or Mueller's method (the second `root` function – a function in **Mathcad** that finds the roots of an equation when root bracketing is not used). Either root-finding method determines the root of  $\mathbf{D}(\mathbf{x})$ . The result of dividing this root by five is `qD4`. Both `root` functions are given because one may not work when the other one does. If the first `root` function fails, type `qD4` on the left side of the equal sign in Equation 3:

$$= (\text{root} [ |P3(\text{seed1}) - (1 - \text{alphaMRUCL})| , \text{seed1}]) / 5 \quad (3)$$

The subprogram `Zseed1` begins by generating values for  $Z_0$  and  $Z_1$ .  $A_0$  and  $A_1$  result from evaluating  $P3(\mathbf{z})$  at  $Z_0$  and  $Z_1$ , respectively. The `while` loop continually increments  $Z_0$  and  $Z_1$  by 5.0 and evaluates  $P3(\mathbf{z})$  at these two values until  $A_1$  becomes greater than `1-alphaMRUCL` for the first time, at which point  $A_0$  will be less than `1-alphaMRUCL`. When this

occurs,  $P3(z)$  is equal to  $1-\text{alphaMRUCL}$  for some value  $z$  between  $Z_0$  and  $Z_1$ . An initial guess for this value is determined using `linterp` (a function in **Mathcad** that performs linear interpolation) and stored in `Zguess`. The initial guess is passed out of the subprogram as `seed1`.

#### 2.4. Page 4

Page 4 of the program is section 4.1. The code in this section is used to determine `qD3`, the `alphaMRLCL` percentage point of the distribution of the studentized range  $Q = w/s$  for subgroup size two with  $\nu$  degrees of freedom. The value `qD3` is used to determine the second-stage short-run lower control chart factor for the *MR* chart. The subprogram `Zseed2` generates the value `seed2` that is used to determine an initial value for `qD3`. An improved value for `qD3` is then calculated by determining the root of the equation  $P3(z) - \text{alphaMRLCL}$  using the Secant or Mueller's method with the seed value `seed2` and dividing this root by five.

The ability of the Secant or Mueller's method to calculate a result depends upon the values for `alphaMRLCL` and `m` (Ridder's or Brent's method should not be used). It is not a problem if it does not calculate a result because the initial value for `qD3` and the improved value match to several places to the right of the decimal. This phenomenon is discussed in more detail when the tabulated results of the program are presented later in section 3. The **Monitor Results** area in the bottom right hand corner of section 4.1 indicates how closely the two values for `qD3` match until the `root` routine fails. This will dictate the number of decimal places that can be used to display `qD3` and the second-stage short-run lower control chart factor for the *MR* chart.

The code in the subprogram `Zseed2` that begins with the first line of code and includes the `while` loop and the two `for` loops constructs  $21 \times 1$  vectors `Zv` for  $z$  and `Av` for  $P3(z)$ . The first row of each vector is zero. The `while` loop determines the first value `Z` where  $P3(Z)$  is greater than `alphaMRLCL`. This `Z` and the corresponding value  $P3(Z)$  are stored in the second rows of `Zv` and `Av`, respectively. The two `for` loops generate values for the remaining rows of `Zv` and `Av`. Two different `for` loops are used because  $P3(z)$  may encounter an error for some `i` (`i`: 1, 2, ..., 20). The value for `i` where the error occurs can be skipped using the dual `for` loop construction. When the execution of this section of code is complete,  $P3(z)$  is equal to `alphaMRLCL` for some value  $z$  between  $Zv_0$  and  $Zv_1$ .

The code in the subprogram `Zseed2` that starts in the line where the variable `Zguess` first appears to the last line of the subprogram is derived from [Harter, Clemm, and Guthrie \(1959\)](#). This code searches for and estimates the value  $z$  where  $P3(z)$  is equal to `alphaMRLCL`. `Zguess` is the initial guess for this value  $z$ . It is determined using `linterp`, the  $21 \times 1$  vectors for  $P3(z)$  and  $z$  previously determined, and `alphaMRLCL`. `Aguess`, the estimated value for  $P3(Zguess)$ , is determined using the **Mathcad** functions `lspline` and `interp`. The function `lspline` returns a vector `vs` which `interp` uses to create a cubic, piecewise polynomial that passes through all of the  $(x, y)$  data points, and assumes the resultant spline curve is linear at the endpoints. The function `interp` returns a spline interpolated value of `Av` at a point `Zguess` and stores the result in `Aguess`. The `while` loop first checks if `Aguess` is an accurate estimate (within  $10^{-15}$ ) of `alphaMRLCL`. If so, `Zguess` is passed out of the subprogram as the value `seed2`. If not, `Aguess` and `Zguess` are entered into the second rows of the previously determined vectors `Av` and `Zv`, respectively, if `Aguess` is more than  $10^{-15}$  larger than `alphaMRLCL`. If `Aguess` is more than  $10^{-15}$  smaller than `alphaMRLCL`, `Aguess` and `Zguess` are entered into the first rows

of the vectors  $\mathbf{Av}$  and  $\mathbf{Zv}$ , respectively. New values for  $\mathbf{Zguess}$  and  $\mathbf{Aguess}$  are determined using the same procedure as before and execution is returned to the beginning of the `while` loop.

## 2.5. Page 5

Page 5 of the program contains sections 5.1 and 5.2. These sections correspond to sections 3.1 and 3.2, respectively, described earlier in subsection 2.3. The only difference is that the calculations in sections 5.1 and 5.2 use  $\nu_{prevm}$  instead of  $\nu$ . The calculations are for `qD4prevm`, the `1-alphaMRUCL` percentage point of the distribution of the studentized range  $Q = w/s$  for subgroup size two with  $\nu_{prevm}$  degrees of freedom. The value `qD4prevm` is used to determine the first-stage short-run upper control chart factor for the  $MR$  chart.

## 2.6. Page 6

Page 6 of the program is section 6.1. This section corresponds to section 4.1 described earlier in subsection 2.4. The only difference is that the calculations in section 6.1 use  $\nu_{prevm}$  instead of  $\nu$ . The calculations are for `qD3prevm`, the `alphaMRLCL` percentage point of the distribution of the studentized range  $Q = w/s$  for subgroup size two with  $\nu_{prevm}$  degrees of freedom. The value `qD3prevm` is used to determine the first-stage short-run lower control chart factor for the  $MR$  chart.

## 2.7. Page 7

Page 7 of the program begins with section 7.1. The value `d2starMR` is Equation 4 in [Elam and Case \(2006\)](#). The value `d2starMRprevm` has the same meaning as `d2starMR`, except it is for  $m - 1$  subgroups. Both `d2starMR` and `d2starMRprevm` are used to determine the two-stage short-run control chart factors for the  $(X, MR)$  charts. The function `qt(adj_alpha, nu)` in **Mathcad** determines the critical value `crit_t` for a cumulative area of `adj_alpha` under the Student's  $t$  curve with  $\nu$  degrees of freedom. The value `crit_t` is used to determine first- and second-stage short-run control chart factors for the  $X$  chart. The function `norm(adj_alpha, 0, 1)` in **Mathcad** determines the critical value `crit_z` for a cumulative area of `adj_alpha` under the standard Normal curve. The value `crit_z` is used to determine the conventional control chart constant for the  $X$  chart.

The next part of page 7 is section 7.2 of the program. The following is a description of the equations in section 7.2 of the program and their corresponding equation numbers in [Elam and Case \(2006\)](#):

- E21: the first-stage short-run control chart factor for the  $X$  chart, Equation 9
- E22: the second-stage short-run control chart factor for the  $X$  chart, Equation 8
- E2: the conventional control chart constant for the  $X$  chart, Equation 14 (the equation for E2 is a generalization of the equation for  $E_2$  from [Wheeler \(1995\)](#)'s Tables 3 and 4 to allow for different values of `alphaInd`)
- D41: the first-stage short-run upper control chart factor for the  $MR$  chart, Equation 12
- D42: the second-stage short-run upper control chart factor for the  $MR$  chart, Equation 10



- D4: the  $\alpha$ -based conventional upper control chart constant for the *MR* chart, Equation 15
- D31: the first-stage short-run lower control chart factor for the *MR* chart, Equation 13
- D32: the second-stage short-run lower control chart factor for the *MR* chart, Equation 11
- D3: the  $\alpha$ -based conventional lower control chart constant for the *MR* chart, Equation 16

The last part of page 7 is the output section of the program. The four values entered at the beginning of the program are given. The control chart factors are broken down into first-stage, second-stage, and conventional. The values for  $\nu$ , `d2starMR`,  $\nu_{\text{prevm}}$ , and `d2starMRprevm`, the mean of the distribution of the range  $W = w/\sigma$  for subgroup size two and the variance of the distribution of the mean moving range  $\bar{M}R/\sigma$ , and [Harter, Clemm, and Guthrie \(1959\)](#)'s Table II.2 results for  $n = 2$  (i.e., for subgroup size two) complete the output of the program. To copy results into another software package (like Excel), follow the directions from **Mathcad**'s help menu or highlight a value and copy and paste it into the other software package. When highlighting a value with the mouse arrow, place the arrow in the middle of the value, depress the left mouse button, and drag the arrow to the right. This will ensure just the numerical value of the result is copied and pasted.

### 3. Tabulated results of the computer program

Tables 1, 2, and 3 were generated using the computer program with the following input values:

- `alphaInd=0.0027`, `alphaMRUCL=0.005`, `alphaMRLCL=0.001`
- `m`: 2-20, 25, 30, 50, 75, 100, 150, 200, 250, 300

The values  $\nu$ , `d2starMR`,  $\nu_{\text{prevm}}$ , `d2starMRprevm`, and `d2` are in Table 1. The  $\nu$  and  $\nu_{\text{prevm}}$  values compare favorably to the equivalent degrees of freedom in Table 3 of [Palm and Wheeler \(1990\)](#) and Table 23 in the appendix of [Wheeler \(1995\)](#). The `d2` value compares favorably to the  $d_2$  value for subgroup size two in [Duncan \(1974\)](#)'s Table M and [Wheeler \(1995\)](#)'s Tables 1 and 18.

The values `qD4`, `qD4prevm`, and `wD4`, as well as `qD3`, `qD3prevm`, and `wD3`, are in Table 2. The results in this table compare favorably to [Harter, Clemm, and Guthrie \(1959\)](#)'s Table II.2 results for  $n = 2$ . As explained in subsection 2.4, in the calculations for `qD3` and `qD3prevm`, the ability of the Secant or Mueller's method to calculate a result depends upon the values for `alphaMRLCL` and `m`. For Table 2, the Secant or Mueller's method fails to work for  $m \geq 3$ . As mentioned previously, this is not a serious issue. The reason is that the initial value for `qD3` matches the improved value for `qD3` (before the Secant or Mueller's method fails) to eight places to the right of the decimal.

Values for `E21`, `D41`, `D31`, `E22`, `D42`, `D32`, `E2`, `D4`, and `D3` are in Table 3. The `E2` value compares favorably to the  $E_2$  value for  $n = 2$  in [Wheeler \(1995\)](#)'s Table 4. It should be noted that the values `wD4` and `wD3` in Table 2 and `D4` and `D3` in Table 3 may differ in the ninth or tenth decimal place for different root routines used to calculate `wD4` and `wD3`.

These favorable comparisons validate the program. Consequently, Table 3 results for `m`: 2-10, 15, 20, 25 may be considered corrections to [Pyzdek \(1993\)](#)'s Table 1 for subgroup size one.

m	$\nu$	$d_2^*(MR)$	m	$\nu$	$d_2^*(MR)$
<b>2</b>	1.00000	1.41421	<b>16</b>	9.49655	1.15842
<b>3</b>	1.58682	1.31072	<b>17</b>	10.10245	1.15660
<b>4</b>	2.19944	1.26009	<b>18</b>	10.70825	1.15499
<b>5</b>	2.81212	1.23124	<b>19</b>	11.31397	1.15356
<b>6</b>	3.42328	1.21271	<b>20</b>	11.91962	1.15227
<b>7</b>	4.03312	1.19982	<b>25</b>	14.94711	1.14740
<b>8</b>	4.64196	1.19034	<b>30</b>	17.97377	1.14418
<b>9</b>	5.25006	1.18308	<b>50</b>	30.07712	1.13780
<b>10</b>	5.85761	1.17734	<b>75</b>	45.20381	1.13464
<b>11</b>	6.46473	1.17269	<b>100</b>	60.32965	1.13306
<b>12</b>	7.07152	1.16885	<b>150</b>	90.58051	1.13150
<b>13</b>	7.67805	1.16562	<b>200</b>	120.83094	1.13072
<b>14</b>	8.28438	1.16287	<b>250</b>	151.08121	1.13025
<b>15</b>	8.89053	1.16049	<b>300</b>	181.33139	1.12994
			<b><math>d_2</math></b>	1.1283791671	

Table 1:  $\nu$  (Degrees of freedom) and  $d_2^*(MR)$  (`d2starMR`) values

m	qD4	qD3	m	qD4	qD3
<b>2</b>	180.05956	0.00222	<b>16</b>	5.13700	0.00182
<b>3</b>	34.23460	0.00206	<b>17</b>	5.05126	0.00182
<b>4</b>	16.63594	0.00198	<b>18</b>	4.97717	0.00181
<b>5</b>	11.42447	0.00193	<b>19</b>	4.91251	0.00181
<b>6</b>	9.12057	0.00190	<b>20</b>	4.85560	0.00181
<b>7</b>	7.86303	0.00188	<b>25</b>	4.64991	0.00180
<b>8</b>	7.08300	0.00187	<b>30</b>	4.52154	0.00180
<b>9</b>	6.55624	0.00186	<b>50</b>	4.28392	0.00179
<b>10</b>	6.17842	0.00185	<b>75</b>	4.17390	0.00178
<b>11</b>	5.89503	0.00184	<b>100</b>	4.12094	0.00178
<b>12</b>	5.67501	0.00184	<b>150</b>	4.06929	0.00178
<b>13</b>	5.49947	0.00183	<b>200</b>	4.04394	0.00178
<b>14</b>	5.35628	0.00183	<b>250</b>	4.02888	0.00178
<b>15</b>	5.23734	0.00182	<b>300</b>	4.01890	0.00177
			$\infty$	3.9697452252	0.0017724543

Table 2: Partial re-creation of Table II.2 in [Harter, Clemm, and Guthrie \(1959\)](#)



m	E21	D41	D31	E22	D42	D32
<b>2</b>	117.89184	—	—	204.19466	127.32134	0.00157
<b>3</b>	22.24670	2.95360	0.00235	31.46159	26.11886	0.00157
<b>4</b>	10.72641	3.58790	0.00209	13.84773	13.20218	0.00157
<b>5</b>	7.34996	3.83736	0.00196	9.00182	9.27880	0.00157
<b>6</b>	5.87022	3.89898	0.00188	6.94574	7.52080	0.00157
<b>7</b>	5.06862	3.89368	0.00183	5.85274	6.55349	0.00157
<b>8</b>	4.57470	3.86822	0.00179	5.18723	5.95038	0.00157
<b>9</b>	4.24308	3.83885	0.00177	4.74391	5.54166	0.00157
<b>10</b>	4.00644	3.81088	0.00175	4.42928	5.24776	0.00157
<b>11</b>	3.82972	3.78583	0.00173	4.19525	5.02691	0.00157
<b>12</b>	3.69307	3.76385	0.00171	4.01479	4.85521	0.00157
<b>13</b>	3.58441	3.74470	0.00170	3.87161	4.71806	0.00157
<b>14</b>	3.49606	3.72800	0.00169	3.75537	4.60610	0.00157
<b>15</b>	3.42287	3.71338	0.00168	3.65920	4.51303	0.00157
<b>16</b>	3.36128	3.70053	0.00168	3.57836	4.43448	0.00157
<b>17</b>	3.30877	3.68916	0.00167	3.50948	4.36732	0.00157
<b>18</b>	3.26348	3.67906	0.00166	3.45012	4.30926	0.00157
<b>19</b>	3.22404	3.67004	0.00166	3.39843	4.25857	0.00157
<b>20</b>	3.18937	3.66194	0.00165	3.35304	4.21395	0.00157
<b>25</b>	3.06459	3.63141	0.00164	3.18972	4.05258	0.00157
<b>30</b>	2.98713	3.61141	0.00162	3.08841	3.95179	0.00157
<b>50</b>	2.84471	3.57258	0.00160	2.90218	3.76510	0.00157
<b>75</b>	2.77924	3.55387	0.00159	2.81655	3.67863	0.00157
<b>100</b>	2.74785	3.54471	0.00159	2.77546	3.63699	0.00157
<b>150</b>	2.71730	3.53570	0.00158	2.73548	3.59637	0.00157
<b>200</b>	2.70234	3.53124	0.00158	2.71588	3.57644	0.00157
<b>250</b>	2.69346	3.52859	0.00158	2.70425	3.56460	0.00157
<b>300</b>	2.68758	3.52682	0.00158	2.69655	3.55675	0.00157
$\infty$	2.65866	3.51810	0.00157	2.65866	3.51810	0.00157

Table 3: Two-stage short-run control chart factors

Problems with [Pyzdek \(1993\)](#)'s Table 1 results are detailed in the conclusion section of [Elam and Case \(2006\)](#).

#### 4. Implications of the tabulated results

Values in Table 3 show some interesting properties. As  $m$  increases, the E22 and D42 values converge in a decreasing manner to E2 and D4, respectively. The D32 values also converge in a decreasing manner to D3, though it is not evident from the accuracy shown. This convergence makes sense because more information about the process is at hand for larger  $m$ .

These results have major implications. A common rule of thumb is that 20 to 30 subgroups of size 4 or 5 (i.e., a combined sample of between 80 and 150 data values) are necessary to use conventional control chart constants for constructing control limits. The results in Table

3 indicate that this may be an incorrect rule when applied to constructing  $(X, MR)$  control charts. Consider Figure 1, which is a plot of E22 and E2 for  $\alpha_{Ind}=0.0027$  and  $m$ : 5, 10, 15, 20, 25, 30, 50, 75, 100, 150. It indicates that if one were to construct  $X$  charts using the conventional control chart constant E2 when 150 or less subgroups of size one (i.e., a combined sample of 150 or less data values) are available to estimate the process mean and standard deviation, the upper and lower control limits would not be wide enough, resulting in a higher probability of a false alarm.

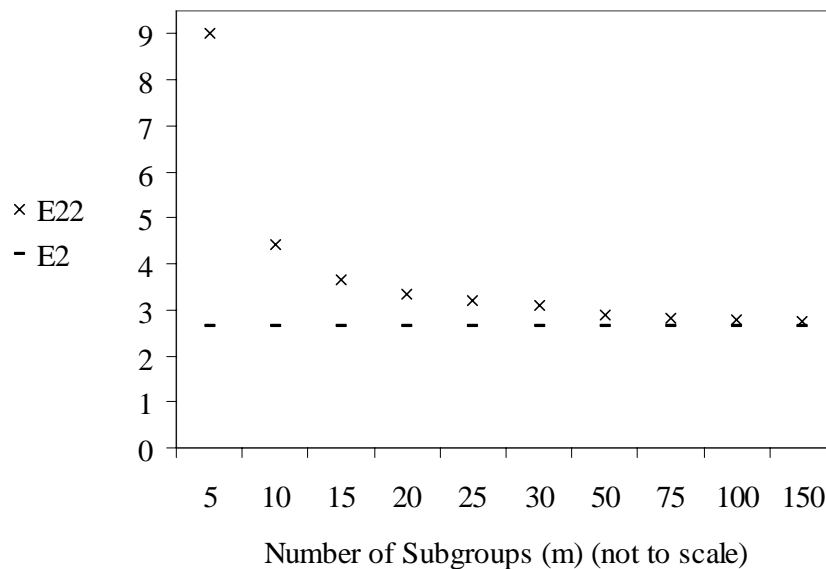
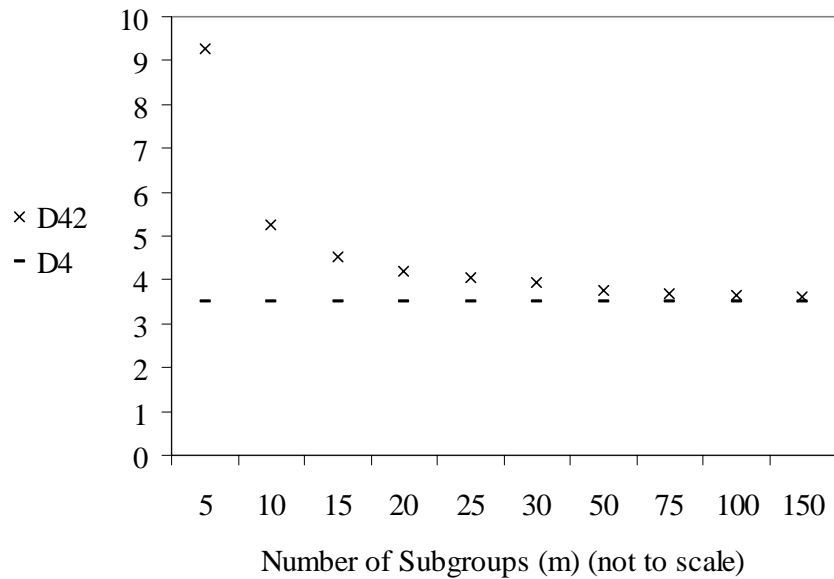


Figure 1: E22 and E2 for  $\alpha_{Ind}=0.0027$

D42 values and D4 in Table 3 also indicate that the common rule of thumb, when applied to constructing  $(X, MR)$  control charts, may be an incorrect rule. Consider Figure 2, which is a plot of D42 and D4 for  $\alpha_{MRUCL}=0.005$  and  $m$ : 5, 10, 15, 20, 25, 30, 50, 75, 100, 150. It indicates that if one were to construct the upper control limit of  $MR$  charts using the conventional control chart constant D4 when 150 or less subgroups of size one are available to estimate the process standard deviation, the upper control limit would not be wide enough, resulting in a higher probability of a false alarm.

To the accuracy shown in Table 3, there is no difference between D32 for any  $m$  and D3. If increased accuracy is used, then D3 is slightly less than D32 for any  $m$ . Consequently, if one were to construct the lower control limit of  $MR$  charts using the conventional control chart constant D3 when 150 or less subgroups of size one are available to estimate the process standard deviation, the lower control limit would be slightly too wide, possibly creating a situation in which the probability of detecting a special cause signal is slightly diminished.

Quesenberry (1993) also investigated the validity of the common rule of thumb when applied to constructing  $(X, MR)$  control charts and concluded that 300 individual values are needed for the  $X$  chart before conventional control chart constants may be used. However, for all practical purposes, the computer program presented by this paper eliminates the need for

Figure 2: D42 and D4 for  $\alpha_{MRUCL}=0.005$ 

these rules.

## 5. A numerical example

Consider the data in Table 4 obtained from a process requiring short-run control charting techniques (assume  $\alpha_{Ind}=0.0027$ ,  $\alpha_{MRUCL}=0.005$ , and  $\alpha_{MRLCL}=0.001$ ).

Subgroup	$\bar{X}$	$MR$
1	1.280	—
2	1.129	0.151
3	1.130	0.001
4	1.131	0.001
5	1.133	0.002
<b>Averages</b>	1.16060	0.03875
<b>Revised Average</b>		0.00133

Table 4: A numerical example

For  $m=5$ , the following first-stage short-run control chart factors for the  $MR$  chart are obtained from Table 3:  $D_{41}=3.83736$  and  $D_{31}=0.00196$ .  $UCL(MR)$  and  $LCL(MR)$  are calculated in Equations 4 and 5, respectively:

$$UCL(MR) = D_{41} \times \bar{MR} = 3.83736 \times 0.03875 = 0.14870 \quad (4)$$

$$LCL(MR) = D_{31} \times \bar{MR} = 0.00196 \times 0.03875 = 0.000076 \quad (5)$$

The first moving range ( $MR=0.151$ ) is above  $UCL(MR)$ . Find, investigate, and remove from the process the special cause (or causes) that created this out of control point, delete the first moving range, recalculate the average moving range (shown as the Revised Average in Table 4), and construct second-stage control limits for the  $(X, MR)$  charts (this approach is from Case (1998)). For  $m=4$ , the following second-stage short-run control chart factors for the  $MR$  chart are obtained from Table 3:  $D42=13.20218$  and  $D32=0.00157$ . For  $m=5$ , the following second-stage short-run control chart factor for the  $X$  chart is obtained from Table 3:  $E22=9.00182$ .  $UCL(MR)$ ,  $LCL(MR)$ ,  $UCL(X)$ , and  $LCL(X)$  are calculated in Equations 6, 7, 8, and 9, respectively:

$$UCL(MR) = D42 \times \bar{MR} = 13.20218 \times 0.00133 = 0.017559 \quad (6)$$

$$LCL(MR) = D32 \times \bar{MR} = 0.00157 \times 0.00133 = 0.0000021 \quad (7)$$

$$UCL(X) = \bar{X} + E22 \times \bar{MR} = 1.16060 + 9.00182 \times 0.00133 = 1.17257 \quad (8)$$

$$LCL(X) = \bar{X} - E22 \times \bar{MR} = 1.16060 - 9.00182 \times 0.00133 = 1.14863 \quad (9)$$

These control limits may be used to monitor the future performance of the process.

## 6. Conclusions

This paper is the second in a series of two papers that fully develops two-stage short-run  $(X, MR)$  control charts. This paper described the development and execution of a computer program that accurately calculates first- and second-stage short-run control chart factors for  $(X, MR)$  charts using the equations derived in the first paper. The computer program makes two important contributions. Those involved with quality control in industry will, for the first time, be able to use theoretically precise control chart factors to determine control limits for  $(X, MR)$  charts regardless of the number of subgroups and  $\alpha$  values. Also, as already mentioned, the computer program eliminates the need for the research question of how many subgroups are enough before conventional control chart constants may be used.

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**Affiliation:**

Matthew E. Elam  
Department of Industrial Engineering  
The University of Alabama  
Box 870288  
Tuscaloosa, AL 35487-0288  
E-mail: [melam@bama.ua.edu](mailto:melam@bama.ua.edu)