



merDeriv: Derivative Computations for Linear Mixed Effects Models with Application to Robust Standard Errors

Ting Wang

The American Board of Anesthesiology

Edgar C. Merkle

University of Missouri

Abstract

While likelihood-based derivatives and related facilities are available in R for many types of statistical models, the facilities are notably lacking for models estimated via **lme4**. This is because the necessary statistical output, including the Hessian, Fisher information and casewise contributions to the model gradient, is not immediately available from **lme4** and is not trivial to obtain. In this article, we describe **merDeriv**, an R package which supplies new functions to obtain analytic output from Gaussian mixed models. We discuss the theoretical results implemented in the code, focusing on calculation of robust standard errors via package **sandwich**. We also use the **sleepstudy** data to illustrate the package and to compare it to a benchmark from package **lavaan**.

Keywords: linear mixed effects model, scores, Huber-White sandwich estimator, robust standard error, **lme4**.

1. Introduction

Package **lme4** (Bates, Mächler, Bolker, and Walker 2015) is widely used to estimate a variety of generalized linear mixed models (GLMMs). Despite its popularity, the package does not provide certain results related to derivatives of the likelihood, which makes it difficult to obtain robust standard errors and other statistical tests. This absence is partially related to the fact that **lme4** does not directly estimate models via likelihood maximization, but rather employs a penalized least squares approach that leads to ML (or REML) estimates (Bates *et al.* 2015). While this approach eases model estimation, it also makes it more difficult to obtain derivatives (first and second) of the likelihood from a fitted model (which are required for, e.g., the Huber-White sandwich estimator). While it is possible to instead utilize the robust estimation methods from package **robustlmm** (Koller 2016), we are interested in directly using

derivative-based methods that rely on estimation of the traditional model. Thus, the goal of this paper is to describe R package **merDeriv** (Wang and Merkle 2018), which is available from the Comprehensive R Archive Network (CRAN) at <https://CRAN.R-project.org/package=merDeriv> and contains functions that compute these derivatives for objects of class ‘**lmerMod**’. We also briefly discuss derivatives associated with models of class ‘**glmerMod**’, though we do not currently have code for these models (the computations are more difficult due to the need for numerical integration).

The paper proceeds as follows. We first describe general notation for the linear mixed model. Next, we derive expressions for the linear mixed models’ casewise (observation level) and clusterwise (cluster/group level) first derivatives, along with the Hessian and Fisher information matrix (including both fixed effect parameters and variances/covariances of random effects). Next, we illustrate the derivatives’ application via the **sleepstudy** data (Belenky *et al.* 2003) included with **lme4**, comparing our results to a benchmark from **lavaan** (Rossee 2012). This illustration includes computation of the Huber-White sandwich estimator (Eicker 1967; White 1980; Huber 1967) for linear mixed models with independent clusters/groups. Finally, we discuss further use and extension of our package’s functionality.

2. Linear mixed model

Following Bates *et al.* (2015), the linear mixed model (LMM) can be written as

$$\mathbf{y}|\mathbf{b} \sim N(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{b}, \mathbf{R}) \quad (1)$$

$$\mathbf{b} \sim N(\mathbf{0}, \mathbf{G}) \quad (2)$$

$$\mathbf{R} = \sigma_r^2 \mathbf{I}_n, \quad (3)$$

where \mathbf{y} is the observed data vector of length n ; \mathbf{X} is an $n \times p$ matrix of fixed covariates; $\boldsymbol{\beta}$ is the fixed effect vector of length p ; \mathbf{Z} is an $n \times q$ design matrix of random effects; and \mathbf{b} is the random effect vector of length q .

The vector \mathbf{b} is assumed to follow a normal distribution with mean $\mathbf{0}$ and covariance matrix \mathbf{G} , where \mathbf{G} is a block diagonal matrix composed of variance/covariance for random effect parameters. The residual covariance matrix, \mathbf{R} , is the product of the residual variance σ_r^2 and an identity matrix of dimension n . We further define $\boldsymbol{\sigma}^2$ to be a vector of length K , containing all variance/covariance parameters (including those of the random effects and the residual). Thus, the matrix \mathbf{G} has $(K - 1)$ unique elements. For example, in a model with two random effects that are allowed to covary, $\boldsymbol{\sigma}^2$ is a vector of length 4 (i.e., $K = 4$). The first three elements correspond to the unique entries of \mathbf{G} , which are commonly expressed as σ_0^2 , σ_{01} , and σ_1^2 . The last component is then the residual variance σ_r^2 .

Based on Equations 1, 2, and 3, the marginal distribution of the LMM is

$$\mathbf{y} \sim N(\mathbf{X}\boldsymbol{\beta}, \mathbf{V}), \quad (4)$$

where

$$\mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}^\top + \sigma_r^2 \mathbf{I}_n. \quad (5)$$

Therefore, the marginal likelihood can be expressed as

$$\ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y}) = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \log(|\mathbf{V}|) - \frac{1}{2} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top \mathbf{V}^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \quad (6)$$

3. Derivative computations for the linear mixed model

In this section, we first discuss analytic results involving the linear mixed model’s first and second derivatives. We then illustrate how these derivatives can be obtained from an object of class ‘`lmerMod`’.

3.1. Scores

Based on the objective function from Equation 6, we derive the score function $s_i()$ for each observation w.r.t. the parameter vector $\boldsymbol{\xi} = (\boldsymbol{\sigma}^2, \boldsymbol{\beta})^\top$. We focus separately on $\boldsymbol{\sigma}^2$ and on $\boldsymbol{\beta}$ below.

Scores for $\boldsymbol{\sigma}^2$

The gradient with respect to the k th entry of $\boldsymbol{\sigma}^2$ ($k = 1, 2, 3, \dots, K$) is (Stroup 2012, p. 136–137):

$$\frac{\partial \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \sigma_k^2} = -\frac{1}{2} \text{tr} \left[\mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \sigma_k^2} \right] + \frac{1}{2} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_k^2} \right) \mathbf{V}^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}), \quad (7)$$

where \mathbf{V} is defined in Equation 5. This gradient sums over i , whereas the scores are defined for each observation i . Thus, to obtain the scores, we can remove the sums from the above equation. This is accomplished by replacing a trace operator with a diag operator, as well as replacing a matrix product with a Hadamard product (also known as elementwise/entrywise multiplication):

$$s(\sigma_k^2; \mathbf{y}) = -\frac{1}{2} \text{diag} \left[\mathbf{V}^{-1} \frac{\partial \mathbf{V}}{\partial \sigma_k^2} \right] + \left\{ \frac{1}{2} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_k^2} \right) \mathbf{V}^{-1} \right\}^\top \circ (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \quad (8)$$

In this way, the gradient of parameter σ_k^2 (a scalar) becomes a $n \times 1$ score vector.

Scores for $\boldsymbol{\beta}$

For the fixed effect parameter $\boldsymbol{\beta}$, the gradient is:

$$\frac{\partial \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\beta}} = \mathbf{X}^\top \mathbf{V}^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \quad (9)$$

The score vector $s(\boldsymbol{\beta}; \mathbf{y})$ can again be obtained by replacing the matrix multiplication by the Hadamard product:

$$s(\boldsymbol{\beta}; \mathbf{y}) = \left\{ \mathbf{X}^\top \mathbf{V}^{-1} \right\}^\top \circ (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \quad (10)$$

The full set of scores can then be expressed as a matrix whose columns consist of the results from Equations 8 and 10.

These equations provide scores for each observation i , and we can construct the clusterwise scores by summing scores within each cluster. In situations with one grouping (clustering) variable, the clusterwise scores can be obtained from our `estfun` method for ‘`lmerMod`’ objects via the default argument `level = 2`. The casewise scores, on the other hand, can be retrieved for all models via the argument `level = 1`.

3.2. Hessian/observed information matrix

The Hessian is the second derivative of the log-likelihood, noted as A^* in this paper. The negative of the Hessian is often called the observed information matrix or observed Fisher information. It is a sample-based version of the Fisher information. Because package **lme4** does not provide a Hessian that includes both the fixed and variance/covariance of random effect (including residual variance) parameters, the derivation of this matrix requires special attention.

To obtain the Hessian, we can divide the matrix A^* into the following four blocks:

$$A^* = \begin{bmatrix} \frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^\top} & \frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\sigma}^2 \top} \\ \frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\sigma}^2 \partial \boldsymbol{\beta}^\top} & \frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\sigma}^2 \partial \boldsymbol{\sigma}^2 \top} \end{bmatrix},$$

where $\boldsymbol{\beta}$ contains all fixed parameters and $\boldsymbol{\sigma}^2$ contains all variance-covariance parameters (in variance-covariance scale) in the linear mixed model. To facilitate the analytic derivations, we index the above four blocks as:

$$A^* = \begin{bmatrix} \text{Block } 1^* & \text{Block } 3^* \\ \text{Block } 2^* & \text{Block } 4^* \end{bmatrix}.$$

Block 1^* is straightforward, which can be obtained by taking the derivative of Equation 9 w.r.t. $\boldsymbol{\beta}$, which can be expressed as:

$$\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^\top} = -\mathbf{X}^\top \mathbf{V}^{-1} \mathbf{X} \quad (11)$$

Derivation of Block 4^* is described in [Stroup \(2012\)](#) and can be written as

$$\begin{aligned} \frac{\partial^2 \ell(\boldsymbol{\sigma}, \mathbf{y}, \boldsymbol{\beta})}{\partial \sigma_{k_1}^2 \partial \sigma_{k_2}^2} &= \left(\frac{1}{2}\right) \text{tr} \left[\mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_1}} \right) \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_2}} \right) \right] \\ &\quad - (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \left\{ \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_1}} \right) \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_2}} \right) \mathbf{V}^{-1} \right\} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}), \end{aligned} \quad (12)$$

where $k_1 \in 1, \dots, K$ and $k_2 \in 1, \dots, K$.

Finally, Block 3^* (which is the transpose of Block 2^*) can be seen as the derivative of Equation 7 w.r.t. $\boldsymbol{\beta}$. Using the identity from [Petersen and Pedersen \(2012, p. 11, Eq. 86\)](#), this allows us to derive Block 3^* as

$$\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\sigma}^2 \partial \boldsymbol{\beta}^\top} = -\mathbf{X}^\top \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \boldsymbol{\sigma}^2} \right) \mathbf{V}^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \quad (13)$$

The results obtained for \mathbf{A}^* are similar to the derivation of the Fisher information matrix, as described below.

3.3. Fisher/expected information matrix

The Fisher information matrix (or expected information matrix) is the expectation of the negative second derivative of the log likelihood, noted as \mathbf{A} throughout the paper. It can often be obtained in R with the help of the `vcov()` function, but package **lme4** only provides results for fixed effect parameters. Thus, we obtain the Fisher information w.r.t. all model parameters by taking the expectation of the negative of the Hessian matrix \mathbf{A}^* .

Specifically, we can express the matrix \mathbf{A} in the following four blocks as before. The only difference is the negative expectation operator.

$$\mathbf{A} = \left[\begin{array}{c|c} -\mathbb{E} \left(\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^\top} \right) & -\mathbb{E} \left(\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\sigma}^2^\top} \right) \\ \hline -\mathbb{E} \left(\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\sigma}^2 \partial \boldsymbol{\beta}^\top} \right) & -\mathbb{E} \left(\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\sigma}^2 \partial \boldsymbol{\sigma}^2^\top} \right) \end{array} \right],$$

Following the same strategy, we index the above four blocks as:

$$\mathbf{A} = \left[\begin{array}{c|c} \text{Block 1} & \text{Block 3} \\ \hline \text{Block 2} & \text{Block 4} \end{array} \right].$$

Because \mathbf{X} and \mathbf{X}^\top are considered constants, Block 1 is simply the negative of Block 1* shown as below:

$$-\mathbb{E} \left(\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^\top} \right) = -\mathbb{E} \left(-\mathbf{X}^\top \mathbf{V}^{-1} \mathbf{X} \right) = \mathbf{X}^\top \mathbf{V}^{-1} \mathbf{X}. \quad (14)$$

This analytic result is mathematically equivalent to the result provided by `solve(vcov())` in **lme4** (which only contains fixed effect parameters).

Derivation of Block 4 is also based on the result from Block 4*. In particular, following the expectation identity for Gaussian distributions from [Petersen and Pedersen \(2012, p. 43, Eq. 380\)](#), the second term of Equation 12 can be transformed to $\text{tr} \left[\mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_1}^2} \right) \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_2}^2} \right) \right]$. Thus Block 4 is reduced to the form shown as below, which is also described in [Stroup \(2012\)](#).

$$-\mathbb{E} \left(\frac{\partial^2 \ell(\boldsymbol{\sigma}, \mathbf{y}, \boldsymbol{\beta})}{\partial \sigma_{k_1}^2 \partial \sigma_{k_2}^2} \right) = \left(\frac{1}{2} \right) \text{tr} \left[\mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_1}^2} \right) \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k_2}^2} \right) \right], \quad (15)$$

where $k_1 \in 1, \dots, K$ and $k_2 \in 1, \dots, K$.

Finally, Block 3 is the negative of the expectation of Block 3*. Using the expectation identity from Petersen and Pedersen (2012, p. 35, Eq. 312), this allows us to derive Block 3 as

$$-\mathbb{E}\left(\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\sigma}^2 \partial \boldsymbol{\beta}^\top}\right) = -\mathbb{E}\left(-\mathbf{X}^\top \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \boldsymbol{\sigma}^2}\right) \mathbf{V}^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right) \quad (16)$$

$$= \mathbf{X}^\top \mathbb{E}\left(\left\{\mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \boldsymbol{\sigma}^2}\right) \mathbf{V}^{-1}\right\} \{(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\}\right) \quad (17)$$

Since $\mathbb{E}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \mathbf{0}$, it leads to $-\mathbb{E}\left(\frac{\partial^2 \ell(\boldsymbol{\sigma}^2, \boldsymbol{\beta}; \mathbf{y})}{\partial \boldsymbol{\sigma}^2 \partial \boldsymbol{\beta}^\top}\right) = \mathbf{0}$, which reflects asymptotic independence of $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}^2$. Thus, we have expressed the necessary derivatives as functions of model matrices and derivatives of the marginal variance \mathbf{V} . We can summarize the Fisher information matrix result for the LMM as:

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{X}^\top \mathbf{V}^{-1} \mathbf{X} & \mathbf{0} \\ \hline \mathbf{0} & \left(\frac{1}{2}\right) \text{tr} \left[\mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k1}^2}\right) \mathbf{V}^{-1} \left(\frac{\partial \mathbf{V}}{\partial \sigma_{k2}^2}\right) \right] \end{array} \right].$$

These results are equivalent to Equations 6.69 to 6.74 of McCulloch and Neuhaus (2001).

We can then invert the information matrix to obtain the variance-covariance matrix. In the `vcov` method for ‘`lmerMod`’ objects from `merDeriv`, we use the argument `full = TRUE` to get the variance-covariance matrix w.r.t. all parameters in the model. If `full = FALSE` (default), the variance-covariance matrix w.r.t. only fixed parameters is returned. To switch between the observed and expected information matrix, we can supply the argument `information = "observed"` or `information = "expected"`. The default option is “expected” due to its wider usage.

4. Relation to ‘`lmerMod`’ objects

In this section, we describe how the quantities needed to compute the scores, Hessian, and Fisher information matrix can be obtained from an ‘`lmerMod`’ object. The data and model matrices \mathbf{y} , \mathbf{X} , $\boldsymbol{\beta}$, and \mathbf{Z} can be obtained directly from `lme4` via `getME()`. The only remaining components, then, are \mathbf{V} and $\partial \mathbf{V} / \partial \boldsymbol{\sigma}^2$. In the following, we focus on how to indirectly obtain these components.

In the `lme4` framework, the random effects covariance matrix \mathbf{G} is decomposed via (Bates *et al.* 2015):

$$\mathbf{G} = \boldsymbol{\Lambda}_\theta \boldsymbol{\Lambda}_\theta^\top \sigma_r^2, \quad (18)$$

where $\boldsymbol{\Lambda}_\theta$ is a $q \times q$ lower diagonal matrix, called the “relative covariance factor”. It can be seen as a Cholesky decomposition of \mathbf{G} / σ_r^2 . The dimension of $\boldsymbol{\Lambda}_\theta$ is the same as that of \mathbf{G} . Additionally, the position of σ_k^2 in \mathbf{G} is the same as the position of θ_k in $\boldsymbol{\Lambda}_\theta$.

Inserting Equation 18 into Equation 5, we can express \mathbf{V} as

$$\mathbf{V} = (\mathbf{Z} \boldsymbol{\Lambda}_\theta \boldsymbol{\Lambda}_\theta^\top \mathbf{Z}^\top + \mathbf{I}_n) \sigma_r^2. \quad (19)$$

Equation 19 is mathematically equivalent to Equation 5, but it has computational advantages when, e.g., a random effect variance is close to 0.

Using Equation 5, the term $\partial\mathbf{V}/\partial\sigma_k^2$ can usually be expressed as

$$\mathbf{Z} \frac{\partial\mathbf{G}}{\partial\sigma_k^2} \mathbf{Z}^\top, \quad (20)$$

so long as σ_k^2 is not the residual variance. The partial derivative $\frac{\partial\mathbf{G}}{\partial\sigma_k^2}$ is then a matrix of the same dimension as \mathbf{G} , with an entry of 1 corresponding to the location of σ_k^2 and 0 elsewhere. Because the location of σ_k^2 within \mathbf{G} matches its location within $\mathbf{\Lambda}_\theta$, we can use $\mathbf{\Lambda}_\theta$ to facilitate computation of $\partial\mathbf{V}/\partial\sigma_k^2$. The only trick is that \mathbf{G} is symmetric, whereas $\mathbf{\Lambda}_\theta$ is lower diagonal. The code below illustrates the implementation of this strategy, where `object` is a fitted model of class `lmerMod`.

```
parts <- getME(object, "ALL")
uluti <- length(parts$theta)
devLambda <- vector("list", uluti)
devV <- vector("list", (uluti + 1))
```

We determine the position of the parameters in the $\mathbf{\Lambda}_\theta$ matrix and we use `forceSymmetric()` to convert the lower diagonal information from $\mathbf{\Lambda}_\theta$ into the symmetric \mathbf{G} .

```
LambdaInd <- parts$Lambda
LambdaInd@x[] <- seq(1:uluti)
for (i in 1:uluti) {
  devLambda[[i]] <- forceSymmetric(LambdaInd == i, uplo = "L")
  devV[[i]] <- tcrossprod(tcrossprod(parts$Z, t(devLambda[[i]])),
    parts$Z)
}
```

Finally, for the derivative with respect to the residual variance, it is obvious that $\partial\mathbf{V}/\partial\sigma_r^2 = \mathbf{I}_n$ so long as $\mathbf{R} = \sigma_r^2\mathbf{I}$ (also see Stroup 2012, p. 137).

The above results are sufficient for obtaining the derivatives necessary for computing the Huber-White sandwich estimator and for carrying out additional statistical tests (see Section 7). In the following sections, we will describe the Huber-White sandwich estimator for linear mixed models with independent clusters, then provide an application.

5. Huber-White sandwich estimator

Let \mathbf{y}_{c_j} contain the observations within cluster c_j . If observations in different clusters are independent (as is the case in many linear mixed models), then we can write

$$\ell(\sigma^2, \boldsymbol{\beta}; \mathbf{y}) = \sum_{j=1}^J \ell(\sigma^2, \boldsymbol{\beta}; \mathbf{y}_{c_j}), \quad (21)$$

where J is the total number of clusters and $\ell(\cdot)$ is defined in Equation 6. The first and second partial derivatives of ℓ w.r.t. $\boldsymbol{\xi} = (\boldsymbol{\sigma}^2 \boldsymbol{\beta})^\top$ can then be written as

$$\ell'(\boldsymbol{\xi}; \mathbf{y}) = \sum_{j=1}^J \frac{\partial \ell(\boldsymbol{\xi}; \mathbf{y}_{c_j})}{\partial \boldsymbol{\xi}} = \sum_{j=1}^J \sum_{i \in c_j} s_i(\boldsymbol{\xi}; y_i), \quad (22)$$

$$\ell''(\boldsymbol{\xi}; \mathbf{y}) = \sum_{j=1}^J \frac{\partial^2 \ell(\boldsymbol{\xi}; \mathbf{y}_{c_j})}{\partial \boldsymbol{\xi}^2}, \quad (23)$$

where $\frac{\partial \ell(\boldsymbol{\xi}; \mathbf{y}_{c_j})}{\partial \boldsymbol{\xi}}$ represents the first derivative within cluster c_j , which can be expressed as the sum of the casewise score $s_i(\cdot)$ belonging to c_j . The function $s_i(\cdot)$ has also been studied in other contexts (e.g., Wang, Merkle, and Zeileis 2014; Zeileis and Hornik 2007).

Inference about $\boldsymbol{\xi}$ relies on a central limit theorem:

$$\sqrt{J}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}) \xrightarrow{d} N(\mathbf{0}, \mathbf{V}(\boldsymbol{\xi})), \quad (24)$$

where \xrightarrow{d} denotes convergence in distribution. The traditional estimate of $\mathbf{V}(\boldsymbol{\xi})$ relies on Equation 23, whereas the Huber-White sandwich estimator of $\mathbf{V}(\boldsymbol{\xi})$ is defined as (e.g., Freedman 2006; White 1980; Zeileis 2006):

$$\mathbf{V}(\hat{\boldsymbol{\xi}}) = (\mathbf{A})^{-1} \mathbf{B} (\mathbf{A})^{-1}, \quad (25)$$

where $\mathbf{A} = -E(\ell''(\hat{\boldsymbol{\xi}}); \mathbf{y})$ and $\mathbf{B} = \text{COV}(\ell'(\hat{\boldsymbol{\xi}}); \mathbf{y})$. The square roots of the diagonal elements of \mathbf{V} are the ‘‘robust standard errors.’’

When the model is correctly specified, the Huber-White sandwich estimator corresponds to the Fisher information matrix. However, the estimator is often used in non-i.i.d. samples to ‘‘correct’’ the information matrix for misspecification (e.g., Freedman 2006). While mixed models explicitly handle lack of independence via random effects, the Huber-White estimators can still be applied to these models to address remaining model misspecifications such as outliers in random effects or deviations from normality (Koller 2016, 2013).

To construct the Huber-White sandwich estimator, \mathbf{A} can be obtained from Equation 23, whose analytic expression for the linear mixed model is expressed in Section 3.2. The matrix \mathbf{B} can then be constructed via (e.g., Freedman 2006):

$$\mathbf{B} = \sum_{j=1}^J \left[\sum_{i \in c_j} s_i(\boldsymbol{\xi}; y_i) \right]^\top \left[\sum_{i \in c_j} s_i(\boldsymbol{\xi}; y_i) \right]. \quad (26)$$

Thus, we require the derivations presented in the previous section: the ‘‘score’’ terms $s_i(\boldsymbol{\xi}; y_i)$ ($i = 1, \dots, n$) and the information matrix using the marginal likelihood from Equation 6.

6. Application

In this section, we illustrate how the package can be used to obtain clusterwise robust standard errors for the `sleepstudy` data (Belenky *et al.* 2003) included in `lme4`. This dataset includes 18 subjects participating in a sleep deprivation study, where each subject’s reaction time was monitored for 10 consecutive days. The reaction times are nested by subject and continuous in measurement, hence the linear mixed model.

We first load package `lme4`, along with the `merDeriv` package that is the focus of this paper.


```
R> library("lme4")
R> library("merDeriv")
```

Next, we fit a model with `Days` as the covariate, including random intercept and slope effects that are allowed to covary. There are six free model parameters: the fixed intercept and slope β_0 and β_1 , the random variance and covariances σ_0^2 , σ_1^2 , and σ_{01} , and the residual variance σ_r^2 .

```
R> lme4fit <- lmer(Reaction ~ Days + (Days|Subject), sleepstudy,
+   REML = FALSE)
```

This particular model can also be estimated as a structural equation model via package `lavaan`, facilitating the comparison of our results with a benchmark. We first convert the data to wide format.

```
R> testwide <- reshape2::dcast(sleepstudy, Subject ~ Days,
+   value.var = "Reaction")
R> names(testwide)[2:11] <- paste("d", 1:10, sep = "")
```

Then we specify the latent model.

```
R> latent <- 'i =~ 1*d1 + 1*d2 + 1*d3 + 1*d4 + 1*d5
+   + 1*d6 + 1*d7 + 1*d8 + 1*d9 + 1*d10
+
+   s =~ 0*d1 + 1*d2 + 2*d3 + 3*d4 + 4*d5
+   + 5*d6 + 6*d7 + 7*d8 + 8*d9 + 9*d10
+
+   d1 ~~ evar*d1
+   d2 ~~ evar*d2
+   d3 ~~ evar*d3
+   d4 ~~ evar*d4
+   d5 ~~ evar*d5
+   d6 ~~ evar*d6
+   d7 ~~ evar*d7
+   d8 ~~ evar*d8
+   d9 ~~ evar*d9
+   d10 ~~ evar*d10
+
+   ## reparameterize as sd
+   sdevar := sqrt(evar)
+   i =~ ivar*i
+   isd := sqrt(ivar)'
```

Finally, we fit the model with `lavaan`.

```
R> lavaanfit <- growth(latent, data = testwide, estimator = "ML")
```

The parameter estimates from the two packages (not shown) all agree to at least two decimal places. Below, we examine the agreement of derivative computations.

Scores

The analytic casewise and clusterwise scores are obtained via the `estfun` method for ‘`lmerMod`’ objects, using the arguments `level = 1` and `level = 2`, respectively. The sum of scores (either casewise or clusterwise) equals the gradient, which is close to zero at the ML estimates.

```
R> score1 <- estfun(lme4fit, level = 1)
R> gradients1 <- colSums(score1)
R> gradients1
```

	(Intercept)	Days
	2.39e-14	2.38e-13
cov_Subject.(Intercept)	cov_Subject.Days.(Intercept)	
	2.94e-09	4.19e-08
cov_Subject.Days		residual
	8.29e-08	-7.38e-09

```
R> score2 <- estfun(lme4fit, level = 2)
R> gradients2 <- colSums(score2)
R> gradients2
```

	(Intercept)	Days
	2.39e-14	2.38e-13
cov_Subject.(Intercept)	cov_Subject.Days.(Intercept)	
	2.94e-09	4.19e-08
cov_Subject.Days		residual
	8.29e-08	-7.38e-09

The clusterwise scores are also provided by a `estfun` method for ‘`lavaan`’ objects in **lavaan**. Figure 1 presents a comparison between the clusterwise scores obtained from the `estfun` methods for ‘`lmerMod`’ and ‘`lavaan`’ objects, showing they are nearly identical. The absolute difference between the scores obtained from these two packages is within 1.5×10^{-7} . The squared differences is within 2.2×10^{-14} .

Variance covariance matrices

We also compare the variance covariance matrix calculated via our **lme4** second derivatives to the `vcov()` output of **lavaan**. The results are displayed in Table 1. The maximum of the absolute difference for all components in the variance covariance matrix is 0.07. This minor difference is due to the fact that **lavaan** applies the delta method to compute the Fisher information matrix for defined parameters (Rosseel 2012; Oberski 2014). In contrast, **merDeriv** utilizes analytic expressions. This difference is ignorable due to the small relative difference (within 10^{-6}).

Finally, the clusterwise Huber-White sandwich estimator is shown in Table 2, which is comparable to the one provided by **lavaan**. The maximum of the absolute difference for all components in the variance covariance matrix is 0.05. The minor difference is again caused by the aforementioned reasons.

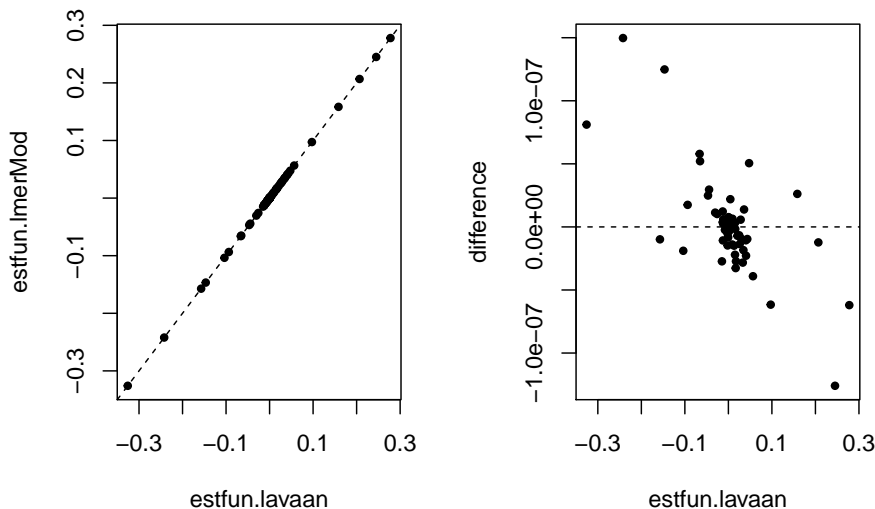


Figure 1: Comparison of scores obtained via the `estfun` methods for ‘`lavaan`’ and ‘`lmerMod`’ objects. In the left panel, the y -axis represents analytic, clusterwise scores obtained from the `estfun` method for ‘`lmerMod`’ objects, and the x -axis represents clusterwise scores obtained from the `estfun` method for ‘`lavaan`’ objects. The dashed line serves as a reference line as $y = x$. In the right panel, the y -axis represents the difference between the scores obtained via the `estfun` methods for ‘`lavaan`’ and ‘`lmerMod`’ objects, and the x -axis represents the clusterwise scores obtained from the `estfun` method for ‘`lavaan`’ objects. The dashed line serves as a reference line as $y = 0$.

7. Discussion

In this paper, we illustrated how to obtain the Huber-White sandwich estimator of estimated parameters arising from objects of class ‘`lmerMod`’ with independent clusters. This required us to derive observational (and clusterwise) scores for fixed and random parameters (leading to the “meat”) as well as a Fisher information matrix that included random effect variances and covariances (leading to the “bread”). In the discussion below, we address extensions to related statistical metrics and models.

7.1. Restricted maximum likelihood (REML)

While we focused on linear mixed models estimated via maximum likelihood (ML), extension to restricted maximum likelihood (REML) is straightforward. The central idea of REML is to maximize the likelihood function of variance parameters after accounting for the fixed effects. By using this approach, the downward bias of ML for variance estimates can be eliminated (similarly to division of n versus $n - 1$ in simple variance calculations), so REML is used often in LMM applications (Stroup 2012).

Referring to the `sleepstudy` example, package `merDeriv` can provide scores (`estfun`) and variance covariance matrix (`vcov`) based on the REML likelihood function and corresponding estimates. The fixed effects parameters are equivalent for ML and REML, whereas the corresponding `vcov` components are larger based on REML. For example, the ML variance for the estimated fixed intercept is 43.99 whereas the REML variance for the estimated fixed intercept is 46.57.

Column name	Row name	merDeriv	lavaan	Abs Diff	Rel Diff
(Intercept)	(Intercept)	43.99	43.99	0.00	0.00
Days	(Intercept)	-1.37	-1.37	0.00	0.00
cov_Subject.(Intercept)	(Intercept)	0.00	0.00	0.00	-
cov_Subject.Days.(Intercept)	(Intercept)	0.00	0.00	0.00	-
cov_Subject.Days	(Intercept)	0.00	0.00	0.00	-
residual	(Intercept)	0.00	0.00	0.00	-
(Intercept)	Days	-1.37	-1.37	0.00	0.00
Days	Days	2.26	2.26	0.00	0.00
cov_Subject.(Intercept)	Days	0.00	0.00	0.00	-
cov_Subject.Days.(Intercept)	Days	0.00	0.00	0.00	-
cov_Subject.Days	Days	0.00	0.00	0.00	-
residual	Days	0.00	0.00	0.00	-
(Intercept)	cov_Subject.(Intercept)	0.00	0.00	0.00	-
Days	cov_Subject.(Intercept)	0.00	0.00	0.00	-
cov_Subject.(Intercept)	cov_Subject.(Intercept)	70366.08	70366.15	0.07	0.00
cov_Subject.Days.(Intercept)	cov_Subject.(Intercept)	-2282.47	-2282.46	0.01	0.00
cov_Subject.Days	cov_Subject.(Intercept)	92.56	92.56	0.00	0.00
residual	cov_Subject.(Intercept)	-2058.08	-2058.08	0.00	0.00
(Intercept)	cov_Subject.Days.(Intercept)	0.00	0.00	0.00	-
Days	cov_Subject.Days.(Intercept)	0.00	0.00	0.00	-
cov_Subject.(Intercept)	cov_Subject.Days.(Intercept)	-2282.47	-2282.46	0.01	0.00
cov_Subject.Days.(Intercept)	cov_Subject.Days.(Intercept)	1838.33	1838.33	0.00	0.00
cov_Subject.Days	cov_Subject.Days.(Intercept)	-115.28	-115.28	0.00	0.00
residual	cov_Subject.Days.(Intercept)	324.96	324.96	0.00	0.00
(Intercept)	cov_Subject.Days	0.00	0.00	0.00	-
Days	cov_Subject.Days	0.00	0.00	0.00	-
cov_Subject.(Intercept)	cov_Subject.Days	92.56	92.56	0.00	0.00
cov_Subject.Days.(Intercept)	cov_Subject.Days	-115.28	-115.28	0.00	0.00
cov_Subject.Days	cov_Subject.Days	184.21	184.21	0.00	0.00
residual	cov_Subject.Days	-72.21	-72.21	0.00	0.00
(Intercept)	residual	0.00	0.00	0.00	-
Days	residual	0.00	0.00	0.00	-
cov_Subject.(Intercept)	residual	-2058.08	-2058.08	0.00	0.00
cov_Subject.Days.(Intercept)	residual	324.96	324.96	0.00	0.00
cov_Subject.Days	residual	-72.21	-72.21	0.00	0.00
residual	residual	5957.61	5957.61	0.00	0.00

Table 1: Comparison between **merDeriv** `vcov` method for ‘`lmerMod`’ objects output and **lavaan** `vcov()` output for the `sleepstudy` data. The first two columns describe the specific matrix entry being compared, the third and fourth columns show the estimates, the fifth and sixth column shows the absolute and relative difference.

7.2. Statistical tests

The scores derived in this paper can potentially be used to carry out a variety of score-based statistical tests. For example, the “fluctuation test” framework discussed by [Zeileis and Hornik \(2007\)](#), [Merkle and Zeileis \(2013\)](#), and others generalizes the traditional score (Lagrange multiplier) test and is used to detect parameter instability across orderings of observations. The tests have been critical for the development of model-based recursive partitioning procedures available via packages such as **partykit** ([Hothorn and Zeileis 2015](#)).

The code that we present here facilitates application of score-based tests to linear mixed models, because the tests described in the previous paragraph are available via object-oriented

Column name	Row name	merDeriv	lavaan	Abs Diff	Rel Diff
(Intercept)	(Intercept)	43.99	43.99	0.00	0.00
Days	(Intercept)	-1.37	-1.37	0.00	0.00
cov_Subject.(Intercept)	(Intercept)	-523.40	-523.41	0.01	0.00
cov_Subject.Days.(Intercept)	(Intercept)	-20.77	-20.77	0.00	0.00
cov_Subject.Days	(Intercept)	-5.92	-5.92	0.00	0.00
residual	(Intercept)	149.15	149.15	0.00	0.00
(Intercept)	Days	-1.37	-1.37	0.00	0.00
Days	Days	2.26	2.26	0.00	0.00
cov_Subject.(Intercept)	Days	-56.09	-56.09	0.00	0.00
cov_Subject.Days.(Intercept)	Days	0.18	0.18	0.00	0.00
cov_Subject.Days	Days	-1.98	-1.98	0.00	0.00
residual	Days	78.71	78.71	0.00	0.00
(Intercept)	cov_Subject.(Intercept)	-523.40	-523.41	0.01	0.00
Days	cov_Subject.(Intercept)	-56.09	-56.09	0.00	0.00
cov_Subject.(Intercept)	cov_Subject.(Intercept)	45232.13	45232.18	0.05	0.00
cov_Subject.Days.(Intercept)	cov_Subject.(Intercept)	1055.38	1055.38	0.00	0.00
cov_Subject.Days	cov_Subject.(Intercept)	427.39	427.39	0.00	0.00
residual	cov_Subject.(Intercept)	-27398.62	-27398.62	0.00	0.00
(Intercept)	cov_Subject.Days.(Intercept)	-20.77	-20.77	0.00	0.00
Days	cov_Subject.Days.(Intercept)	0.18	0.18	0.00	0.00
cov_Subject.(Intercept)	cov_Subject.Days.(Intercept)	1055.38	1055.38	0.00	0.00
cov_Subject.Days.(Intercept)	cov_Subject.Days.(Intercept)	1862.99	1862.99	0.00	0.00
cov_Subject.Days	cov_Subject.Days.(Intercept)	-89.28	-89.28	0.00	0.00
residual	cov_Subject.Days.(Intercept)	1214.37	1214.37	0.00	0.00
(Intercept)	cov_Subject.Days	-5.92	-5.92	0.00	0.00
Days	cov_Subject.Days	-1.98	-1.98	0.00	0.00
cov_Subject.(Intercept)	cov_Subject.Days	427.39	427.39	0.00	0.00
cov_Subject.Days.(Intercept)	cov_Subject.Days	-89.28	-89.28	0.00	0.00
cov_Subject.Days	cov_Subject.Days	137.89	137.89	0.00	0.00
residual	cov_Subject.Days	-492.56	-492.56	0.00	0.00
(Intercept)	residual	149.15	149.15	0.00	0.00
Days	residual	78.71	78.71	0.00	0.00
cov_Subject.(Intercept)	residual	-27398.62	-27398.62	0.00	0.00
cov_Subject.Days.(Intercept)	residual	1214.37	1214.37	0.00	0.00
cov_Subject.Days	residual	-492.56	-492.56	0.00	0.00
residual	residual	43229.03	43229.03	0.00	0.00

Table 2: Comparison of the `sleepstudy` sandwich estimator obtained from our `merDeriv` code with the analogous estimator obtained from `lavaan`. The first two columns describe the specific matrix entry being compared, the third and fourth columns show the estimates, the fifth and sixth column shows the absolute and relative difference.

R packages. That is to say the aforementioned packages can be applied to linear mixed models estimated via `lme4`, because we have supplied the generic function `estfun` for models of class ‘`lmerMod`’. A challenge involves the fact that much of the above theory requires observations to be independent. For the linear mixed models with independent clusters, tests can often be applied immediately. However, while we can test parameter instability across independent clusters, it is more difficult to test for instability across correlated observations within a cluster. A related issue, further described below, arises when we attempt to apply sandwich estimators to models with crossed random effects.

7.3. Models with multiple random effects terms

The “independence” challenges described in the previous section translate to the setting of models with multiple random effects terms, such as (partially) crossed random effects or models with multilevel nested designs (e.g., three-level models; Bates 2005, Chapter 2). These correspond to situations where there are at least two unique variables defining clusters (for example, clusters defined by primary school attended and by secondary school attended). In this case, we cannot simply sum scores within a cluster to obtain independent, clusterwise scores. This is because observations in different clusters on the first grouping variable may be in the same cluster on the second grouping variable. Thus, it is unclear how the statistical machinery developed for independent observations (e.g., robust standard errors, instability tests) can transfer to models with partially crossed random effects. While our `estfun` method for ‘`lmerMod`’ objects can return casewise scores and the `vcov` method for ‘`lmerMod`’ objects can return the full variance covariance matrix of all ‘`lmerMod`’ objects, it is unclear how to further use these results.

The main difficulty involves construction of the “meat”, which is the variance of the first derivatives based on the grouping variable. One possible solution is to create separate “meats” based on different grouping variables, accounting for covariances between the meats. This approach is described in Rasbash and Goldstein (1994) and Cameron, Gelbach, and Miller (2011) to decompose parameter variances when there are multiple grouping variables. It may be possible to apply the same idea to our problem, and we plan to study this in the future.

7.4. GLMM

Finally, the procedures described here for scores, Hessians, Fisher information and sandwich estimators can be extended to generalized linear mixed models estimated via `glmer()`. The technical difficulty involved with this extension is the observational scores. In the linear mixed model, we can derive the analytic scores for each observation because we know that the marginal distribution is normal. In the GLMM, the marginal distribution is typically unknown, and we require integral approximation methods (e.g., quadrature or the Laplace approximation) to obtain the scores and second derivatives. Combination of these integral approximation methods with the `lme4` penalized least squares approach presents a challenge that we have not yet overcome. We plan to do so in the future.

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References

- Bates D, Mächler M, Bolker B, Walker S (2015). “Fitting Linear Mixed-Effects Models Using `lme4`.” *Journal of Statistical Software*, **67**(1), 1–48. doi:10.18637/jss.v067.i01.
- Bates DM (2005). “`lme4`: Mixed-Effects Modeling with R.” Chapter drafts, URL <http://lme4.R-Forge.R-project.org/lmmwR/>.

- Belenky G, Wesensten NJ, Thorne DR, Thomas ML, Sing HC, Redmond DP, Russo MB, Balkin TJ (2003). “Patterns of Performance Degradation and Restoration During Sleep Restriction and Subsequent Recovery: A Sleep Dose-Response Study.” *Journal of Sleep Research*, **12**(1), 1–12. doi:10.1046/j.1365-2869.2003.00337.x.
- Cameron AC, Gelbach JB, Miller DL (2011). “Robust Inference with Multiway Clustering.” *Journal of Business & Economic Statistics*, **29**(2), 238–249. doi:10.1198/jbes.2010.07136.
- Eicker F (1967). “Limit Theorems for Regressions with Unequal and Dependent Errors.” In LM LeCam, J Neyman (eds.), *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pp. 59–82. University of California Press, Berkeley.
- Freedman DA (2006). “On the So-Called “Huber Sandwich Estimator” and “Robust Standard Errors”.” *The American Statistician*, **60**(4), 299–302. doi:10.1198/000313006x152207.
- Hothorn T, Zeileis A (2015). “**partykit**: A Modular Toolkit for Recursive Partytioning in R.” *Journal of Machine Learning Research*, **16**, 3905–3909.
- Huber PJ (1967). “The Behavior of Maximum Likelihood Estimates Under Nonstandard Conditions.” In LM LeCam, J Neyman (eds.), *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pp. 221–233. University of California Press, Berkeley.
- Koller M (2013). *Robust Estimation of Linear Mixed Models*. Ph.D. thesis, ETH Zürich, Switzerland.
- Koller M (2016). “**robustlmm**: An R Package for Robust Estimation of Linear Mixed-Effects Models.” *Journal of Statistical Software*, **75**(6), 1–24. doi:10.18637/jss.v075.i06.
- McCulloch CE, Neuhaus JM (2001). *Generalized Linear Mixed Models*. John Wiley & Sons.
- Merkle EC, Zeileis A (2013). “Tests of Measurement Invariance without Subgroups: A Generalization of Classical Methods.” *Psychometrika*, **78**(1), 59–82. doi:10.1007/s11336-012-9302-4.
- Oberski DL (2014). “**lavaan.survey**: An R Package for Complex Survey Analysis of Structural Equation Models.” *Journal of Statistical Software*, **57**(1), 1–27. doi:10.18637/jss.v057.i01.
- Petersen KB, Pedersen MS (2012). “The Matrix Cookbook.” Version 20121115, URL <http://www2.imm.dtu.dk/pubdb/p.php?3274>.
- Rasbash J, Goldstein H (1994). “Efficient Analysis of Mixed Hierarchical and Cross-Classified Random Structures Using a Multilevel Model.” *Journal of Educational and Behavioral Statistics*, **19**(4), 337–350.
- Rosseel Y (2012). “**lavaan**: An R Package for Structural Equation Modeling.” *Journal of Statistical Software*, **48**(2), 1–36. doi:10.18637/jss.v048.i02.

- Stroup WW (2012). *Generalized Linear Mixed Models: Modern Concepts, Methods and Applications*. Chapman & Hall/CRC, New York.
- Wang T, Merkle E (2018). *merDeriv: Case-Wise and Cluster-Wise Derivatives for Mixed Effects Models*. R package version 0.1-6, URL <https://CRAN.R-project.org/package=merDeriv>.
- Wang T, Merkle E, Zeileis A (2014). “Score-Based Tests of Measurement Invariance: Use in Practice.” *Frontiers in Psychology*, **5**(438), 1–11. doi:10.3389/fpsyg.2014.00438.
- White H (1980). “A Heteroskedasticity-Consistent Covariance Matrix Estimator and a Direct Test for Heteroskedasticity.” *Econometrica: Journal of the Econometric Society*, **48**(4), 817–838. doi:10.2307/1912934.
- Zeileis A (2006). “Object-Oriented Computation of Sandwich Estimators.” *Journal of Statistical Software*, **16**(9), 1–16. doi:10.18637/jss.v016.i09.
- Zeileis A, Hornik K (2007). “Generalized M-Fluctuation Tests for Parameter Instability.” *Statistica Neerlandica*, **61**(4), 488–508. doi:10.1111/j.1467-9574.2007.00371.x.

Affiliation:

Ting Wang
The American Board of Anesthesiology
4208 Six Forks Road, Suite 1500
Raleigh, NC 27609-5765, United States of America
E-mail: twb8d@mail.missouri.edu

Edgar C. Merkle
Department of Psychological Sciences
University of Missouri
28A McAlester Hall
Columbia, MO 65211, United States of America
E-mail: merklee@missouri.edu
URL: <http://faculty.missouri.edu/~merklee/>